

THE c -MAP, TITS SATAKE SUBALGEBRAS AND THE SEARCH FOR $\mathcal{N} = 2$ INFLATON POTENTIALS

P. FRÉ^{a1}, A.S. SORIN^b AND M. TRIGIANTE^c

^a*Dipartimento di Fisica, Università di Torino
INFN – Sezione di Torino
via P. Giuria 1, 10125 Torino ITALY
e-mail: fre@to.infn.it*

^b*Bogoliubov Laboratory of Theoretical Physics & Laboratory of High Energy Physics,
Joint Institute for Nuclear Research,
141980 Dubna, Moscow Region, Russia
e-mail: sorin@theor.jinr.ru*

^c*Dipartimento di Fisica Politecnico di Torino,
C.so Duca degli Abruzzi, 24, I-10129 Torino, Italy
e-mail: mario.trigiant@gmail.com*

Abstract

In this paper we address the general problem of including inflationary models exhibiting Starobinsky-like potentials into (symmetric) $\mathcal{N} = 2$ supergravities. This is done by gauging suitable abelian isometries of the hypermultiplet sector and then truncating the resulting theory to a single scalar field. By using the characteristic properties of the global symmetry groups of the $\mathcal{N} = 2$ supergravities we are able to make a general statement on the possible α -attractor models which can be obtained upon truncation. We find that in symmetric $\mathcal{N} = 2$ models group theoretical constraints restrict the allowed values of the parameter α to be $\alpha = 1, \frac{2}{3}, \frac{1}{3}$. This confirms and generalizes results recently obtained in the literature. Our analysis heavily relies on the mathematical structure of symmetric $\mathcal{N} = 2$ supergravities, in particular on the so called c -map connection between Quaternionic Kähler manifolds starting from Special Kähler ones. A general statement on the possible consistent truncations of the gauged models, leading to Starobinsky-like potentials, requires the essential help of Tits Satake universality classes. The paper is mathematically self-contained and aims at presenting the involved mathematical structures to a public not only of physicists but also of mathematicians. To this end the main mathematical structures and the general gauging procedure of $\mathcal{N} = 2$ supergravities is reviewed in some detail.

¹Prof. Fré is presently fulfilling the duties of Scientific Counselor of the Italian Embassy in the Russian Federation, Denezhnij pereulok, 5, 121002 Moscow, Russia.

Contents

I	INTRODUCING THE SUBJECT AND MOTIVATIONS	4
1	INTRODUCTION	5
II	MATHEMATICAL THEORY OF THE c-MAP	9
2	SPECIAL KÄHLER GEOMETRY	9
2.1	HODGE–KÄHLER MANIFOLDS	9
2.2	CONNECTION ON THE LINE BUNDLE	10
2.3	SPECIAL KÄHLER MANIFOLDS	11
2.4	THE VECTOR KINETIC MATRIX $\mathcal{N}_{\Lambda\Sigma}$ IN SPECIAL GEOMETRY	13
2.5	THE HOLOMORPHIC MOMENT MAP ON KÄHLER MANIFOLDS	14
3	QUATERNIONIC GEOMETRY	17
3.1	QUATERNIONIC KÄHLER, VERSUS HYPERKÄHLER MANIFOLDS	18
3.2	THE TRIHOLOMORPHIC MOMENT MAP ON QUATERNIONIC MANIFOLDS	20
4	THE QUATERNIONIC KÄHLER GEOMETRY IN THE IMAGE OF THE c-MAP	22
4.1	THE HYPERKÄHLER TWO-FORMS AND THE $\mathfrak{su}(2)$ -CONNECTION	23
4.2	ISOMETRIES OF \mathcal{QM} IN THE IMAGE OF THE c -MAP AND THEIR TRI-HOLOMORPHIC MOMENT MAPS	25
4.2.1	TRI-HOLOMORPHIC MOMENT MAPS FOR THE HEISENBERG ALGEBRA TRANSLATIONS .	26
4.2.2	TRI-HOLOMORPHIC MOMENT MAP FOR THE HEISENBERG ALGEBRA CENTRAL CHARGE	26
4.2.3	TRI-HOLOMORPHIC MOMENT MAP FOR THE EXTENSION OF \mathcal{SK}_n HOLOMORPHIC ISOMETRIES	27
4.3	HOMOGENEOUS SYMMETRIC SPECIAL QUATERNIONIC KÄHLER MANIFOLDS	28
4.3.1	THE TRI-HOLOMORPHIC MOMENT MAP IN HOMOGENEOUS SYMMETRIC QUATERNIONIC KÄHLER MANIFOLDS	32
III	ABELIAN GAUGINGS AND GENERAL PROPERTIES OF THEIR POTENTIALS IN THE c-MAP FRAMEWORK	34
5	MINIMAL COUPLING SPECIAL GEOMETRY	35
5.1	GAUGING ABELIAN ISOMETRIES OF THE HYPERMULTIPLETS	37
5.2	THE STAROBINSKY POTENTIAL	37
IV	EXAMPLES	40

6	THE S^3 MODEL AND ITS QUATERNIONIC IMAGE $\frac{G_{(2,2)}}{SU(2) \times SU(2)}$	40
6.1	THE SPECIAL KÄHLER STRUCTURE OF S^3	40
6.2	THE MATRIX \mathcal{M}_4^{-1} AND THE c -MAP	41
6.3	THE VIELBEIN AND THE BORELLIAN MAURER CARTAN EQUATIONS	42
6.4	THE SPIN CONNECTION	43
6.5	HOLONOMY ALGEBRA AND DECOMPOSITION OF THE SPIN CONNECTION	44
6.6	STRUCTURE OF THE ISOTROPY SUBALGEBRA \mathbb{H}	45
6.7	THE COSET REPRESENTATIVE	46
7	THE $Sp(6, \mathbb{R})/SU(3) \times U(1)$ - MODEL AND ITS C-MAP IMAGE.	48
7.1	THE $\mathfrak{sp}(6, \mathbb{R})$ LIE ALGEBRA	50
7.2	THE REPRESENTATION $14'$	52
7.3	THE HOLOMORPHIC SYMPLECTIC SECTION AND ITS TRANSFORMATION IN THE $14'$	52
7.4	THE KÄHLER POTENTIAL AND THE METRIC	54
7.4.1	THE SOLVABLE PARAMETRIZATION	55
7.5	THE QUARTIC INVARIANT IN THE $14'$	56
7.6	TRUNCATION TO THE STU -MODEL	56
7.7	REDUCTION OF THE CHARGE VECTOR TO THE $(2, 2, 2)$	58
7.7.1	FURTHER REDUCTION TO NORMAL FORM OF THE CHARGE VECTOR	59
8	THE $\frac{F_{(4,4)}}{SU(2) \times USp(6)}$ QUATERNIONIC KÄHLER MANIFOLD	59
8.1	THE MAXIMAL COMPACT SUBALGEBRA $\mathbb{H} = \mathfrak{su}(2) \oplus \mathfrak{usp}(6)$	61
8.1.1	THE $\mathfrak{su}(2)$ LIE ALGEBRA	61
8.1.2	THE $\mathfrak{usp}(6)$ LIE ALGEBRA	61
8.2	THE SUBALGEBRA $\mathfrak{sl}(2, \mathbb{R})_E \oplus \mathfrak{sp}(6, \mathbb{R})$ AND THE \mathbf{W} -GENERATORS	62
8.2.1	THE EHLERS SUBALGEBRA $\mathfrak{sl}(2, \mathbb{R})_E$	63
8.2.2	THE SUBALGEBRA $\mathfrak{sp}(6, \mathbb{R})$	63
8.2.3	THE \mathbf{W} -GENERATORS	64
8.3	THE SOLVABLE COSET REPRESENTATIVE	64
8.4	THE EXAMPLE OF THE INCLUSION OF MULTI STAROBINSKY MODELS	65
8.5	NILPOTENT GAUGINGS AND TRUNCATIONS	67
8.5.1	CONCLUSION OF THE ABOVE DISCUSSION	74
V	CONCLUSIONS AND PERSPECTIVES	76
9	THE TITS SATAKE PROJECTION	76
10	TITS SATAKE UNIVERSALITY CLASSES AND THE EMBEDDING OF STAROBINSKY-LIKE MODELS	77
10.1	UNIVERSALITY CLASS $\mathfrak{sp}(6, \mathbb{R}) \Rightarrow \mathfrak{f}_{4(4)}$	79
10.1.1	$\mathfrak{su}(3, 3)$ MODEL	79
10.1.2	$\mathfrak{so}^*(12)$ MODEL	79
10.1.3	$\mathfrak{e}_{7(-25)}$ MODEL	80
10.2	UNIVERSALITY CLASS $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2, 3) \Rightarrow \mathfrak{so}(4, 5)$	80
10.3	\mathbf{W} -REPRESENTATIONS OF THE MAXIMALLY SPLIT NON EXOTIC MODELS	81

10.4 GAUGINGS WITH CONSISTENT ONE-FIELD TRUNCATIONS	84
11 CONCLUSIONS	84
A COSET REPRESENTATIVES AND OTHER LARGE FORMULAE NOT DISPLAYED IN THE MAIN TEXT	86
A.1 CONCERNING THE $G_{(2,2)}$ MODEL	86
A.2 EXPLICIT FORM OF GENERATORS FOR THE LIE ALGEBRA $\mathfrak{sp}(6, \mathbb{R})$ IN THE $\mathbf{14}'$ REPRESENTATION	86
A.2.1 T	93

Part I

INTRODUCING THE SUBJECT AND MOTIVATIONS

The present one is a research paper and it contains some new original results. These latter are mostly of mathematical-geometrical character and in our opinion they might be of some interest both for the mathematical scientific community, as well as for that of the theoretical physicists working in the field of supergravity/superstrings. The motivations for the present study is that of analyzing within a general geometric framework some of the recent results [1] relative to the inclusion of candidate inflaton potentials into extended supergravity, the aim being that of singling out general mathematical patterns that eventually can be exported to other examples that include more fields and more multiplets.

From the physicist's viewpoint the obvious ultimate goal is that of singling out possible chains of inclusions of the inflationary models into hierarchically larger and more (super)-symmetric theories finally sheading light on the appropriate place of the inflaton-dynamics, which is revealed by observational cosmology, within a duely unified theory of all interactions.

Notwithstanding the above mentioned specific motivations, the geometrical results presented in this paper, have an intrinsic mathematical interest and moreover admit different applications than those in cosmology within the very framework of supergravities and their gaugings. For this reason we have chosen to make this paper self-consistent and readable by a wider audience of both mathematicians and physicists working in fields different from that of supergravity cosmological models, by presenting in a systematic way all the mathematical definitions and structures that we utilize in the original part of our work. In the same spirit, in order to provide our reader with orientation, the paper is subdivided into Parts and sections.

The present first part provides a conceptual introduction and the physical motivations in relation to current research.

The second part provides the definitions of Kähler-Hodge, Special Kähler and Quaternionic Kähler manifolds. Then introduces the c -map [2] and illustrates how in Quaternionic Kähler manifolds lying in the image of c -map, all the quaternionic structures, in particular the HyperKähler two-forms, the $\mathfrak{su}(2)$ -connection and the tri-holomorphic moment maps of isometries can all be constructed purely in terms of Special geometry data.

The third part discusses abelian gaugings of hypermultiplet isometries in $\mathcal{N} = 2$ supergravity. Using the general mathematical formulae derived in Part Two we discuss generic properties and features of the ensuing scalar potentials.

The fourth part presents concrete examples. In the particular case of the c -map of the S^3 model we retrieve the results obtained in [1]. Another relevant Special Kähler Geometry that we consider corresponds to that of the symmetric space $\mathrm{Sp}(6, \mathbb{R})/\mathrm{SU}(3) \times \mathrm{U}(1)$. For this model we provide an in depth, full fledged construction. Analyzing its Quaternionic Kähler extension by means of the c -map we are able to generalize the results of [1] showing that they fall into a general pattern. Our detailed construction may have applications both in the cosmological perspective of the present paper and in the classification of Black-Hole solutions, possibly also in other contexts.

Part five summarizes the results obtained in the two considered examples and shows that they unveil a deep and universal structure. We briefly summarize the organization of $\mathcal{N} = 2$ scalar manifolds that are symmetric spaces into Tits–Satake universality classes [3]. Such a concept already proved to be of high value in relation to the construction and classification of supergravity black hole solutions [4]. It is equally effective and precious in relation to the c -map and cosmological models. Indeed relying on these structures we are able to show that the inclusion of Starobinsky-like models into extended supergravity theories has a universal character being

associated with the gauging of the universal sub-Tits-Satake subalgebra $\mathbb{G}_{\text{subTS}} = \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2)$. This also leads to the prediction that available values of α for the so named α -attractors [5] are just $\alpha = 1, \frac{2}{3}, \frac{1}{3}$. After this conceptual elaboration, part five contains our conclusions and remarks on further perspectives.

1 INTRODUCTION

The recent observational results on the power-spectrum of the Cosmological Microwave Background radiation [6],[7],[8],[9] have stirred renewed interest in one-field inflationary cosmological models [10],[12],[13],[14]. Indeed the type of cosmology [15],[16] that seems to be consistent with observations is that based on the simplest scenario of just one scalar field ϕ (*the inflaton*) minimally coupled to Einstein Gravity and endowed with a suitable scalar potential $V(\phi)$.

In view of this, several studies have been devoted to the problem of including into supergravity potentials $V(\phi)$ that produce an early inflationary phase and have good cosmological properties, [17],[18],[19],[21],[22],[23],[24],[25],[26],[5],[27]. A separate investigation was also devoted to determine a list of one field integrable potentials [28] and to discuss their possible inclusion into $\mathcal{N} = 1$ supergravity by means of suitable superpotentials [29]. It was shown that such type of inclusion is quite difficult and can be realized only in very few cases [29].

Notwithstanding the difficulties with the inclusion of inflaton potentials by means of a superpotential (F-terms), approximately one year ago, in the seminal paper [30] it was instead pointed out that every positive-definite potential $V(\phi)$ can be minimally included into $\mathcal{N} = 1$ supergravity as a D-term (see [31], [32] and [33] for the complete structure of matter coupled $\mathcal{N} = 1$ supergravity). It suffices to introduce a Kähler one-fold Σ with an abelian group of isometries G_{iso} and a Kähler potential \mathcal{K} related to the potential V by an appropriate differential relation which allows to interpret this latter as the square of the moment-map of the holomorphic Killing vector generating G_{iso} . From the mathematical point of view this new vision stimulated the formulation of the concept of D -map and its extensive study in [33] and [34]. The essential mechanism behind the D -map is the just the Brout-Englert-Higgs mechanism as it is realized in supergravity. Gauging an isometry of the Kählerian scalar manifold \mathcal{M}_K one generates both mass terms for the fermions and a scalar potential that is the square of the moment map \mathcal{P}_k of the corresponding Killing vector. The gauge vector field A_μ utilized to gauge the considered isometry becomes massive by eating up one of the two scalar fields composing the Wess-Zumino multiplet, while its partner remains in the Lagrangian as a degree of freedom of spin zero, self interacting by means of the D -term potential [35],[36].

Among the positive definite scalar potentials that can be included in supergravity in this way there are the Starobinsky-like potentials [10]:

$$V_{\text{Starobinsky-like}} = \text{const} \times \left(1 - \exp \left[-\sqrt{\frac{2}{3\alpha}} \phi \right] \right)^2. \quad (1.1)$$

When $\alpha = 1$, the potential (1.1) emerges in the second derivative supergravity dual of a higher derivative $R + R^2$ supergravity model [11], [37],[38],[39],[40]. Due to the high relevance of these potentials in phenomenology, several studies were devoted to the mechanisms for their inclusion in supergravity². In their minimal D -term realization, the Starobinsky-like potentials were shown to be generated by the gauging of the parabolic

²The inclusion of the original Starobinsky $R + R^2$ model in $\mathcal{N} = 2$ supergravity was discussed in [1] (see [20] for an earlier discussion) where it is shown to be dual to an $\mathcal{N} = 2$ model with two long massive vector multiplets on a $\mathcal{N} = 2$ Minkowski vacuum. The scalaron is subject to a scalar potential of the form (1.1) with $\alpha = 1$. The models considered here are not dual to the $R + R^2$ Starobinsky model and share with it only the scalaron potential, which we shall refer to as the *Starobinsky potential*. If $\alpha \neq 1$ the potential (1.1) will be called *Starobinsky-like*.

subgroups of $SL(2, \mathbb{R})$ in a theory where the Kähler one-fold Σ is the homogeneous space $\frac{SL(2, \mathbb{R})}{SO(2)}$, with a value of the curvature directly related to α by:

$$R_\alpha = -\frac{2}{3\alpha} \quad (1.2)$$

The case $\alpha = 1$ which is the proper Starobinsky potential corresponds to a realization of the $\frac{SL(2, \mathbb{R})}{SO(2)}$ geometry which is not only Kählerian but actually Special Kählerian. Indeed it is the case of the S^3 model.

In more general terms the minimal supergravity approach and the D -map from potentials to Kähler geometry posed the question of relating the type of generated potential with the global topology of the isometry whose gauging generates it. This issue was thoroughly studied in [41],[42]. Relying on notions developed by Gromov et al [43],[44], in [41],[42] it was shown that the global topology of the isometry group can be characterized as, *elliptic, parabolic or hyperbolic* on general Kähler manifolds Σ of non-positive curvature and that the three cases lead to different distinctive properties of the moment-maps, that either have a fixed-point at finite distance (elliptic case) or a fixed point on the boundary (parabolic) or two fixed point on the boundary (hyperbolic). In the case of constant negative curvature Kähler one-folds, *i.e.* of $\Sigma_\alpha = \frac{SL(2, \mathbb{R})}{SO(2)}$ manifolds, it was shown in [41] and [42] that one generates the following three potentials:

$$V(\phi) = \begin{cases} V_{elliptic} &= \left(\cosh \left[\sqrt{\frac{2}{3\alpha}} \phi \right] - \kappa \right)^2 &\Leftrightarrow \text{from gauging of an elliptic subgroup} \\ V_{hyperbolic} &= \left(\sinh \left[\sqrt{\frac{2}{3\alpha}} \phi \right] - \kappa \right)^2 &\Leftrightarrow \text{from gauging of a hyperbolic subgroup} \\ V_{parabolic} &= \left(\exp \left[-\sqrt{\frac{2}{3\alpha}} \phi \right] - \kappa \right)^2 &\Leftrightarrow \text{from gauging of a parabolic subgroup} \end{cases} \quad (1.3)$$

where α parameterizes the curvature of the one-fold Σ_α according to eq. (1.2) and the parameter $\kappa = \pm 1, 0$ is interpreted as the coupling constant of a Fayet Iliopoulos term [45]. Utilizing instead a flat Kähler one-fold $\Sigma_{flat} = \mathbb{C}$, it was shown in [41] and [42] that the gauging of an elliptic isometry yields a mexican hat Brout-Englert-Higgs potential, while the quadratic potential of chaotic inflation is generated by a translation parabolic gauging.

In view of the above conclusions an obvious and very much relevant question concerns the possible inclusion of the potentials (1.3) into extended supergravity, in particular $\mathcal{N} = 2$. Such a question amounts to asking whether there are consistent one-field truncations of appropriate gauged extended supergravities that produce the considered potentials. Such a question includes two issues:

1. the choice of a gauging,
2. the existence of an appropriate consistent truncation.

The first of these two issues is relevant for another important problem, that of constructing and classifying de Sitter or Minkowskian, stable or metastable, vacua in supergravity and it was extensively addressed in that context, [46],[47],[48],[49], [50],[51]. Let us spend some words on the second issue.

The general form of the scalar field equations in supergravity is that of a σ -model with a potential, namely the following one:

$$0 = \square \phi^I + \Gamma_{JK}^I(\phi) \partial_\mu \phi^J \partial^\mu \phi^K + \frac{\partial}{\partial \phi^I} V(\phi) \quad (1.4)$$

where the scalars ϕ^I are coordinates of the target Riemannian manifold \mathcal{M}_{scalar} , by $\Gamma_{JK}^I(\phi)$ we have denoted the Levi-Civita connection on the former and V is the potential. The problem of consistent truncation boils

down to the following. We consider the embedding of a Σ_α Kählerian one-fold into the scalar manifold:

$$\pi : \Sigma_\alpha \mapsto \mathcal{M}_{scalar} \quad (1.5)$$

and we require that the pull-back of the field equations (1.4) onto the surface Σ_α :

$$0 = \phi^\star \left[\square \phi^I + \Gamma_{JK}^I(\phi) \partial_\mu \phi^J \partial^\mu \phi^K + \frac{\partial}{\partial \phi^I} V(\phi) \right] \quad (1.6)$$

be consistent, namely that it reproduces always the same equations for all values of I . What are the restrictions on the embedding π that guarantee such a consistency? This question can be answered in a general form when \mathcal{M}_{scalar} is a symmetric homogeneous space G/H , as it happens most frequently in supergravity models. Considering as usual the symmetric decomposition of the Lie algebra \mathbb{G} :

$$\begin{aligned} \mathbb{G} &= \mathbb{H} \oplus \mathbb{K} \\ [\mathbb{H}, \mathbb{H}] &\subset \mathbb{H} \\ [\mathbb{H}, \mathbb{K}] &\subset \mathbb{K} \\ [\mathbb{K}, \mathbb{K}] &\subset \mathbb{H} \end{aligned} \quad (1.7)$$

the embedding (1.5) induces a homomorphism:

$$\pi : \mathfrak{sl}(2, \mathbb{R}) \mapsto \mathbb{G} \quad ; \quad \pi : \mathbb{O}(2) \mapsto \mathbb{H} \quad ; \quad \pi : \mathbb{K}_2 \mapsto \mathbb{K} \quad (1.8)$$

where:

$$\mathfrak{sl}(2, \mathbb{R}) = \mathbb{O}(2) \oplus \mathbb{K}_2 \quad (1.9)$$

Let us introduce the centralizer algebra of the image of $\mathbb{O}(2)$ in \mathbb{H} , namely

$$\forall \mathfrak{g} \in \mathbb{H} \quad : \quad \mathfrak{g} \in \mathbb{N}_\pi \quad \text{iff} \quad [\mathfrak{g}, \pi(\mathbb{O}(2))] = 0 \quad (1.10)$$

The subspace \mathbb{K} decomposes into irreducible representations of $\pi(\mathbb{O}(2)) \oplus \mathbb{N}_\pi$. The embedding π leads to a consistent truncation if such a decomposition has the following structure:

$$\mathbb{K} = \underbrace{(\mathbf{2}_1 | \mathbf{1})}_{\pi(\mathbb{K}_2)} \oplus_{i=1}^m (\mathbf{2}_{q_i} | \mathbf{D}_i) \oplus (\mathbf{1} | \mathbf{D}_0) \quad (1.11)$$

where $\mathbf{2}_q$ denotes a doublet representation of $\pi(\mathbb{O}(2))$ with charge q and all \mathbf{D}_i and \mathbf{D}_0 are transitive representations, namely carry a *color* of \mathbb{N}_π , the only \mathbb{N}_π -singlet being the subspace $\pi(\mathbb{K}_2)$ tangent to the embedded one-fold Σ_α . In this case setting to zero all the colored fields (those not in $\pi(\mathbb{K}_2)$) is consistent because the equation of a colored field cannot receive contribution from two colorless singlets.

In the case of $\mathcal{N} = 2$ supergravity the scalar manifold is actually the direct product of two manifolds: a special Kähler manifold \mathcal{M}_{SK} that contains the vector multiplet scalars and a Quaternionic Kähler \mathcal{M}_Q manifold describing the hypermultiplet scalars. Hence one comes to the question whether the one-fold Σ_α associated with the inflaton is to be embedded in \mathcal{M}_{SK} or in \mathcal{M}_Q . The significant advance introduced by [1] is the scenario in which Σ_α goes into \mathcal{M}_Q and its abelian isometry is gauged by means of vector multiplets assigned to the so named Minimal Coupling Special Geometry. This choice allows for a generic stabilization

of the vector multiplet scalars and allows to focus only on the properties of the Quaternionic Kähler manifold \mathcal{M}_Q . In [1] the authors considered the cases where

$$\mathcal{M}_Q = \frac{G_{(2,2)}}{SU(2) \times SU(2)} \text{ and } \mathcal{M}_Q = \frac{SO(1,4)}{SO(4)}, \quad (1.12)$$

and it was shown how all the potentials (1.3) can be included with possible values of $\alpha = 1, \frac{2}{3}, \frac{1}{3}$ (the value $\alpha = 2/3$ could be obtained only for the quaternionic projective space). In the present paper we address the same question for generic homogeneous symmetric Quaternionic Kähler manifolds and we eventually arrive at the same conclusion. The three potentials (1.3) can always be embedded with the same possible values of α . Our main weapons in reaching such a conclusion and proving its generality are two:

- a)** The c -map from Special Kähler manifolds \mathcal{M}_{SK} to Quaternionic Kähler manifolds \mathcal{M}_Q
- b)** The Tits-Satake projection and the Tits-Satake universality classes.

Combining these two mathematical instruments we are able to look inside the hypermultiplet manifold and to single out its inner core which is the Special Kähler STU-model. Working in this framework we were able to derive simple general formulae for the triholomorphic moment maps which besides their present use in cosmology might admit several other interesting and useful applications. Similarly the Tits Satake structure allows for a deep understanding of the universal character of the Starobinsky like potentials and opens the way to their further uplift to higher \mathcal{N} theories in the perspective of finding *microscopic interpretations* of the gaugings that generates them.

Part II

MATHEMATICAL THEORY OF THE c -MAP

As announced in the introductory part, the aim of this section of the paper is two-fold. On the one hand we want to review the general geometric structure of homogeneous symmetric Quaternionic Kähler manifolds which are in the image of the c -map and to present in a unified fashion general analytic formulae for the complex structures, $\mathfrak{su}(2)$ -connection and tri-holomorphic moment maps. The ultimate goal is the use of such mathematical instruments in the quest of retrieving *inflaton potentials* inside properly gauged $\mathcal{N} = 2$ supergravity theories. On the other hand we aim at a concise, yet comprehensive presentation of Special Kähler geometry, Quaternionic Kähler geometry and the c -map which might be readable by mathematicians, in particular differential geometers and Lie algebrists. Indeed this mathematical subject was mostly developed by theoretical physicists and it is not widely known in the mathematical community. This is rather unfortunate, especially in the light of its relevance to the issue of the classification of nilpotent orbits, which, instead, is largely explored by mathematicians, and is quite relevant both to the classification of black hole solutions and emerges now as quite important in cosmological issues. Unfortunately the mathematical results on nilpotent orbits are obtained within frameworks that make no reference to the very particular structures of special geometry, quaternionic geometry and the magic relations of the c -map. Spreading awareness of this sophisticated and beautiful mathematics among mathematicians might be beneficial to both communities and it is the second aim of the following sections. A further, humbler, yet quite important aim, pursued here, is that of establishing some unified and systematic notations that might be utilized in subsequent publications devoted to a systematic exploration of the reach field of *gaugings*, *consistent truncations*, *de Sitter vacua* and *inflaton potentials*.

2 SPECIAL KÄHLER GEOMETRY

Special Kähler geometry in special coordinates was introduced in 1984–85 by B. de Wit et al. and E. Cremmer et al. (see pioneering papers in [52]), where the coupling of $\mathcal{N} = 2$ vector multiplets to $\mathcal{N} = 2$ supergravity was fully determined. The more intrinsic definition of special Kähler geometry in terms of symplectic bundles is due to Strominger (1990), who obtained it in connection with the moduli spaces of Calabi–Yau compactifications, (see [53]). The coordinate-independent description and derivation of special Kähler geometry in the context of $\mathcal{N} = 2$ supergravity is due to Castellani, D’Auria, Ferrara [54] and to D’Auria, Ferrara, Fre’ (1991)[55]. Homogenous symmetric special Kähler manifolds were classified before by Cremmer and Van Proyen in [56]. An early review in modern mathematical language is provided by [57]. The structure of isometry group for both Special Kähler and Quaternionic Kähler manifolds was extensively studied in [58],[61]

Let us summarize the relevant concepts and definitions

2.1 HODGE–KÄHLER MANIFOLDS

Consider a *line bundle* $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$ over a Kähler manifold \mathcal{M} . By definition this is a holomorphic vector bundle of rank $r = 1$. For such bundles the only available Chern class is the first:

$$c_1(\mathcal{L}) = \frac{i}{2} \bar{\partial} (h^{-1} \partial h) = \frac{i}{2} \bar{\partial} \partial \log h \quad (2.1)$$

where the 1-component real function $h(z, \bar{z})$ is some hermitian fibre metric on \mathcal{L} . Let $\xi(z)$ be a holomorphic section of the line bundle \mathcal{L} : noting that under the action of the operator $\bar{\partial} \partial$ the term $\log(\bar{\xi}(\bar{z}) \xi(z))$ yields a vanishing contribution, we conclude that the formula in eq.(2.1) for the first Chern class can be re-expressed as follows:

$$c_1(\mathcal{L}) = \frac{i}{2} \bar{\partial} \partial \log \|\xi(z)\|^2 \quad (2.2)$$

where $\|\xi(z)\|^2 = h(z, \bar{z}) \bar{\xi}(\bar{z}) \xi(z)$ denotes the norm of the holomorphic section $\xi(z)$.

Eq.(2.2) is the starting point for the definition of Hodge–Kähler manifolds. A Kähler manifold \mathcal{M} is a Hodge manifold if and only if there exists a line bundle $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$ such that its first Chern class equals the cohomology class of the Kähler two-form K :

$$c_1(\mathcal{L}) = [K] \quad (2.3)$$

In local terms this means that there is a holomorphic section $\xi(z)$ such that we can write

$$K = \frac{i}{2} g_{ij^*} dz^i \wedge d\bar{z}^{j^*} = \frac{i}{2} \bar{\partial} \partial \log \|\xi(z)\|^2 \quad (2.4)$$

Recalling the local expression of the Kähler metric in terms of the Kähler potential $g_{ij^*} = \partial_i \partial_{j^*} \mathcal{K}(z, \bar{z})$, it follows from eq.(2.4) that if the manifold \mathcal{M} is a Hodge manifold, then the exponential of the Kähler potential can be interpreted as the metric $h(z, \bar{z}) = \exp(\mathcal{K}(z, \bar{z}))$ on an appropriate line bundle \mathcal{L} .

2.2 CONNECTION ON THE LINE BUNDLE

On any complex line bundle \mathcal{L} there is a canonical hermitian connection defined as :

$$\theta \equiv h^{-1} \partial h = \frac{1}{h} \partial_i h dz^i \quad ; \quad \bar{\theta} \equiv h^{-1} \bar{\partial} h = \frac{1}{h} \partial_{i^*} h d\bar{z}^{i^*} \quad (2.5)$$

For the line-bundle advocated by the Hodge–Kähler structure we have

$$[\bar{\partial} \theta] = c_1(\mathcal{L}) = [K] \quad (2.6)$$

and since the fibre metric h can be identified with the exponential of the Kähler potential we obtain:

$$\theta = \partial \mathcal{K} = \partial_i \mathcal{K} dz^i \quad ; \quad \bar{\theta} = \bar{\partial} \mathcal{K} = \partial_{i^*} \mathcal{K} d\bar{z}^{i^*} \quad (2.7)$$

To define special Kähler geometry, in addition to the afore-mentioned line-bundle \mathcal{L} we need a flat holomorphic vector bundle $\mathcal{SV} \rightarrow \mathcal{M}$ whose sections play an important role in the construction of the supergravity Lagrangians. For reasons intrinsic to such constructions the rank of the vector bundle \mathcal{SV} must be $2n_V$ where n_V is the total number of vector fields in the theory. If we have n -vector multiplets the total number of vectors is $n_V = n + 1$ since, in addition to the vectors of the vector multiplets, we always have the graviphoton sitting in the graviton multiplet. On the other hand the total number of scalars is $2n$. Suitably paired into n -complex fields z^i , these scalars span the n complex dimensions of the base manifold \mathcal{M} to the rank $2n + 2$ bundle $\mathcal{SV} \rightarrow \mathcal{M}$.

In the sequel we make extensive use of covariant derivatives with respect to the canonical connection of the line-bundle \mathcal{L} . Let us review its normalization. As it is well known there exists a correspondence between line-bundles and $U(1)$ -bundles. If $\exp[f_{\alpha\beta}(z)]$ is the transition function between two local trivializations of the line-bundle $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$, the transition function in the corresponding principal $U(1)$ -bundle $\mathcal{U} \rightarrow \mathcal{M}$ is just

$\exp[i\text{Im}f_{\alpha\beta}(z)]$ and the Kähler potentials in two different charts are related by: $\mathcal{K}_\beta = \mathcal{K}_\alpha + f_{\alpha\beta} + \bar{f}_{\alpha\beta}$. At the level of connections this correspondence is formulated by setting: $\text{U}(1)$ -connection $\equiv \mathcal{Q} = \text{Im}\theta = -\frac{i}{2}(\theta - \bar{\theta})$. If we apply this formula to the case of the $\text{U}(1)$ -bundle $\mathcal{U} \rightarrow \mathcal{M}$ associated with the line-bundle \mathcal{L} whose first Chern class equals the Kähler class, we get:

$$\mathcal{Q} = \frac{i}{2} \left(\partial_i \mathcal{K} dz^i - \partial_{i^*} \mathcal{K} d\bar{z}^{i^*} \right) \quad (2.8)$$

Let now $\Phi(z, \bar{z})$ be a section of \mathcal{U}^p . By definition its covariant derivative is $\nabla\Phi = (d - ip\mathcal{Q})\Phi$ or, in components,

$$\nabla_i \Phi = (\partial_i + \frac{1}{2}p\partial_i \mathcal{K})\Phi \quad ; \quad \nabla_{i^*} \Phi = (\partial_{i^*} - \frac{1}{2}p\partial_{i^*} \mathcal{K})\Phi \quad (2.9)$$

A covariantly holomorphic section of \mathcal{U} is defined by the equation: $\nabla_{i^*} \Phi = 0$. We can easily map each section $\Phi(z, \bar{z})$ of \mathcal{U}^p into a section of the line-bundle \mathcal{L} by setting:

$$\tilde{\Phi} = e^{-p\mathcal{K}/2} \Phi. \quad (2.10)$$

With this position we obtain:

$$\nabla_i \tilde{\Phi} = (\partial_i + p\partial_i \mathcal{K})\tilde{\Phi} \quad ; \quad \nabla_{i^*} \tilde{\Phi} = \partial_{i^*} \tilde{\Phi} \quad (2.11)$$

Under the map of eq.(2.10) covariantly holomorphic sections of \mathcal{U} flow into holomorphic sections of \mathcal{L} and viceversa.

2.3 SPECIAL KÄHLER MANIFOLDS

We are now ready to give the first of two equivalent definitions of special Kähler manifolds:

Definition 2.1 *A Hodge Kähler manifold is **Special Kähler (of the local type)** if there exists a completely symmetric holomorphic 3-index section W_{ijk} of $(T^*\mathcal{M})^3 \otimes \mathcal{L}^2$ (and its antiholomorphic conjugate $W_{i^*j^*k^*}$) such that the following identity is satisfied by the Riemann tensor of the Levi-Civita connection:*

$$\begin{aligned} \partial_{m^*} W_{ijk} &= 0 & \partial_m W_{i^*j^*k^*} &= 0 \\ \nabla_{[m} W_{i]jk} &= 0 & \nabla_{[m} W_{i^*]j^*k^*} &= 0 \\ \mathcal{R}_{i^*j\ell^*k} &= g_{\ell^*j} g_{ki^*} + g_{\ell^*k} g_{ji^*} - e^{2\mathcal{K}} W_{i^*\ell^*s^*} W_{tkj} g^{s^*t} \end{aligned} \quad (2.12)$$

In the above equations ∇ denotes the covariant derivative with respect to both the Levi-Civita and the $\text{U}(1)$ holomorphic connection of eq.(2.8). In the case of W_{ijk} , the $\text{U}(1)$ weight is $p = 2$.

Out of the W_{ijk} we can construct covariantly holomorphic sections of weight 2 and - 2 by setting:

$$C_{ijk} = W_{ijk} e^{\mathcal{K}} \quad ; \quad C_{i^*j^*k^*} = W_{i^*j^*k^*} e^{\mathcal{K}} \quad (2.13)$$

The flat bundle mentioned in the previous subsection apparently does not appear in this definition of special geometry. Yet it is there. It is indeed the essential ingredient in the second definition whose equivalence to the first we shall shortly provide.

Let $\mathcal{L} \xrightarrow{\pi} \mathcal{M}$ denote the complex line bundle whose first Chern class equals the cohomology class of the Kähler form K of an n -dimensional Hodge-Kähler manifold \mathcal{M} . Let $\mathcal{SV} \rightarrow \mathcal{M}$ denote a holomorphic

flat vector bundle of rank $2n + 2$ with structural group $\text{Sp}(2n + 2, \mathbb{R})$. Consider tensor bundles of the type $\mathcal{H} = \mathcal{SV} \otimes \mathcal{L}$. A typical holomorphic section of such a bundle will be denoted by Ω and will have the following structure:

$$\Omega = \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \quad \Lambda, \Sigma = 0, 1, \dots, n \quad (2.14)$$

By definition the transition functions between two local trivializations $U_i \subset \mathcal{M}$ and $U_j \subset \mathcal{M}$ of the bundle \mathcal{H} have the following form:

$$\begin{pmatrix} X \\ F \end{pmatrix}_i = e^{f_{ij}} M_{ij} \begin{pmatrix} X \\ F \end{pmatrix}_j \quad (2.15)$$

where f_{ij} are holomorphic maps $U_i \cap U_j \rightarrow \mathbb{C}$ while M_{ij} is a constant $\text{Sp}(2n + 2, \mathbb{R})$ matrix. For a consistent definition of the bundle the transition functions are obviously subject to the cocycle condition on a triple overlap: $e^{f_{ij} + f_{jk} + f_{ki}} = 1$ and $M_{ij} M_{jk} M_{ki} = 1$.

Let $i\langle \cdot | \cdot \rangle$ be the compatible hermitian metric on \mathcal{H}

$$i\langle \Omega | \bar{\Omega} \rangle \equiv -i\Omega^T \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \bar{\Omega} \quad (2.16)$$

Definition 2.2 *We say that a Hodge–Kähler manifold \mathcal{M} is **special Kähler** if there exists a bundle \mathcal{H} of the type described above such that for some section $\Omega \in \Gamma(\mathcal{H}, \mathcal{M})$ the Kähler two form is given by:*

$$K = \frac{i}{2} \partial \bar{\partial} \log (i\langle \Omega | \bar{\Omega} \rangle) = \frac{i}{2} g_{i,j^*} dz^i \wedge d\bar{z}^{j^*} \dots \quad (2.17)$$

From the point of view of local properties, eq.(2.17) implies that we have an expression for the Kähler potential in terms of the holomorphic section Ω :

$$\mathcal{K} = -\log (i\langle \Omega | \bar{\Omega} \rangle) = -\log [i (\bar{X}^\Lambda F_\Lambda - \bar{F}_\Sigma X^\Sigma)] \quad (2.18)$$

The relation between the two definitions of special manifolds is obtained by introducing a non-holomorphic section of the bundle \mathcal{H} according to:

$$V = \begin{pmatrix} L^\Lambda \\ M_\Sigma \end{pmatrix} \equiv e^{\mathcal{K}/2} \Omega = e^{\mathcal{K}/2} \begin{pmatrix} X^\Lambda \\ F_\Sigma \end{pmatrix} \quad (2.19)$$

so that eq.(2.18) becomes:

$$1 = i\langle V | \bar{V} \rangle = i (\bar{L}^\Lambda M_\Lambda - \bar{M}_\Sigma L^\Sigma) \quad (2.20)$$

Since V is related to a holomorphic section by eq.(2.19) it immediately follows that:

$$\nabla_{i^*} V = \left(\partial_{i^*} - \frac{1}{2} \partial_{i^*} \mathcal{K} \right) V = 0 \quad (2.21)$$

On the other hand, from eq.(2.20), defining:

$$\begin{aligned} U_i &= \nabla_i V = \left(\partial_i + \frac{1}{2} \partial_i \mathcal{K} \right) V \equiv \begin{pmatrix} f_i^\Lambda \\ h_{\Sigma|i} \end{pmatrix} \\ \bar{U}_{i^*} &= \nabla_{i^*} \bar{V} = \left(\partial_{i^*} + \frac{1}{2} \partial_{i^*} \mathcal{K} \right) \bar{V} \equiv \begin{pmatrix} \bar{f}_{i^*}^\Lambda \\ \bar{h}_{\Sigma|i^*} \end{pmatrix} \end{aligned}$$

it follows that:

$$\nabla_i U_j = i C_{ijk} g^{k\ell^*} \bar{U}_{\ell^*} \quad (2.22)$$

where ∇_i denotes the covariant derivative containing both the Levi-Civita connection on the bundle \mathcal{TM} and the canonical connection θ on the line bundle \mathcal{L} . In eq.(2.22) the symbol C_{ijk} denotes a covariantly holomorphic ($\nabla_{\ell^*} C_{ijk} = 0$) section of the bundle $\mathcal{TM}^3 \otimes \mathcal{L}^2$ that is totally symmetric in its indices. This tensor can be identified with the tensor of eq.(2.13) appearing in eq.(2.12). Alternatively, the set of differential equations:

$$\nabla_i V = U_i \quad (2.23)$$

$$\nabla_i U_j = i C_{ijk} g^{k\ell^*} \bar{U}_{\ell^*} \quad (2.24)$$

$$\nabla_{i^*} U_j = g_{i^*j} V \quad (2.25)$$

$$\nabla_{i^*} V = 0 \quad (2.26)$$

with V satisfying eq.s (2.19, 2.20) give yet another definition of special geometry. In particular it is easy to find eq.(2.12) as integrability conditions of(2.26)

2.4 THE VECTOR KINETIC MATRIX $\mathcal{N}_{\Lambda\Sigma}$ IN SPECIAL GEOMETRY

In the construction of supergravity actions another essential item is the complex symmetric matrix $\mathcal{N}_{\Lambda\Sigma}$ whose real and imaginary parts are necessary in order to write the kinetic terms of the vector fields. The matrix $\mathcal{N}_{\Lambda\Sigma}$ constitutes an integral part of the Special Geometry set up and we provide its general definition in the following lines. Explicitly $\mathcal{N}_{\Lambda\Sigma}$ which, in relation to its interpretation in the case of Calabi-Yau threefolds, is named the *period matrix*, is defined by means of the following relations:

$$\bar{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma} \bar{L}^\Sigma \quad ; \quad h_{\Sigma|i} = \mathcal{N}_{\Lambda\Sigma} f_i^\Lambda \quad (2.27)$$

which can be solved introducing the two $(n+1) \times (n+1)$ vectors

$$f_I^\Lambda = \begin{pmatrix} f_i^\Lambda \\ \bar{L}^\Lambda \end{pmatrix} \quad ; \quad h_{\Lambda|I} = \begin{pmatrix} h_{\Lambda|i} \\ \bar{M}_\Lambda \end{pmatrix} \quad (2.28)$$

and setting:

$$\mathcal{N}_{\Lambda\Sigma} = h_{\Lambda|I} \circ (f^{-1})^I_\Sigma \quad (2.29)$$

Let us now consider the case where the Special Kähler manifold \mathcal{SK}_n of complex dimension n has some isometry group $U_{\mathcal{SK}}$. Compatibility with the Special Geometry structure requires the existence of a $2n+2$ -dimensional symplectic representation of such a group that we name the **W** representation. In other words that there necessarily exists a symplectic embedding of the isometry group \mathcal{SK}_n

$$U_{\mathcal{SK}} \mapsto \text{Sp}(2n+2, \mathbb{R}) \quad (2.30)$$

such that for each element $\xi \in \mathcal{U}_{\mathcal{SK}}$ we have its representation by means of a suitable real symplectic matrix:

$$\xi \mapsto \Lambda_\xi \equiv \begin{pmatrix} A_\xi & B_\xi \\ C_\xi & D_\xi \end{pmatrix} \quad (2.31)$$

satisfying the defining relation (in terms of the symplectic antisymmetric metric \mathbb{C}):

$$\Lambda_\xi^T \underbrace{\begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{1}_{n \times n} \\ -\mathbf{1}_{n \times n} & \mathbf{0}_{n \times n} \end{pmatrix}}_{\equiv \mathbb{C}} \Lambda_\xi = \underbrace{\begin{pmatrix} \mathbf{0}_{n \times n} & \mathbf{1}_{n \times n} \\ -\mathbf{1}_{n \times n} & \mathbf{0}_{n \times n} \end{pmatrix}}_{\mathbb{C}} \quad (2.32)$$

which implies the following relations on the $n \times n$ blocks:

$$\begin{aligned} A_\xi^T C_\xi - C_\xi^T A_\xi &= 0 \\ A_\xi^T D_\xi - C_\xi^T B_\xi &= \mathbf{1} \\ B_\xi^T C_\xi - D_\xi^T A_\xi &= -\mathbf{1} \\ B_\xi^T D_\xi - D_\xi^T B_\xi &= 0 \end{aligned} \quad (2.33)$$

Under an element of the isometry group the symplectic section Ω of Special Geometry transforms as follows:

$$\Omega(\xi \cdot z) = \Lambda_\xi \Omega(z) \quad (2.34)$$

As a consequence of its definition, under the same isometry the matrix \mathcal{N} transforms by means of a generalized linear fractional transformation:

$$\mathcal{N}(\xi \cdot z, \xi \cdot \bar{z}) = (C_\xi + D_\xi \mathcal{N}(z, \bar{z})) (A_\xi + B_\xi \mathcal{N}(z, \bar{z}))^{-1} \quad (2.35)$$

2.5 THE HOLOMORPHIC MOMENT MAP ON KÄHLER MANIFOLDS

The concept of holomorphic moment map applies to all Kähler manifolds, not necessarily special. Indeed it can be constructed just in terms of the Kähler potential without advocating any further structure. In this subsection we review its properties and definition, as usual in order to fix conventions, normalizations and notations.

Let $g_{ij\star}$ be the Kähler metric of a Kähler manifold \mathcal{M} and let us assume that $g_{ij\star}$ admits a non trivial group of continuous isometries \mathcal{G} generated by Killing vectors $k_{\mathbf{I}}^i$ ($\mathbf{I} = 1, \dots, \dim \mathcal{G}$) that define the infinitesimal variation of the complex coordinates z^i under the group action:

$$z^i \rightarrow z^i + \epsilon^{\mathbf{I}} k_{\mathbf{I}}^i(z) \quad (2.36)$$

Let $k_{\mathbf{I}}^i(z)$ be a basis of holomorphic Killing vectors for the metric $g_{ij\star}$. Holomorphicity means the following differential constraint:

$$\partial_{j\star} k_{\mathbf{I}}^i(z) = 0 \leftrightarrow \partial_j k_{\mathbf{I}}^{i\star}(\bar{z}) = 0 \quad (2.37)$$

while the generic Killing equation (suppressing the gauge index \mathbf{I}):

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0 \quad (2.38)$$

in holomorphic indices reads as follows:

$$\nabla_i k_j + \nabla_j k_i = 0 \quad ; \quad \nabla_{i^*} k_j + \nabla_j k_{i^*} = 0 \quad (2.39)$$

where the covariant components are defined as $k_j = g_{ji^*} k_{i^*}$ (and similarly for k_{i^*}).

The vectors $k_{\mathbf{I}}^i$ are generators of infinitesimal holomorphic coordinate transformations $\delta z^i = \epsilon^{\mathbf{I}} k_{\mathbf{I}}^i(z)$ which leave the metric invariant. In the same way as the metric is the derivative of a more fundamental object, the Killing vectors in a Kähler manifold are the derivatives of suitable prepotentials. Indeed the first of eq.s (2.39) is automatically satisfied by holomorphic vectors and the second equation reduces to the following one:

$$k_{\mathbf{I}}^i = i g^{ij^*} \partial_{j^*} \mathcal{P}_{\mathbf{I}}, \quad \mathcal{P}_{\mathbf{I}}^* = \mathcal{P}_{\mathbf{I}} \quad (2.40)$$

In other words if we can find a real function $\mathcal{P}^{\mathbf{I}}$ such that the expression $i g^{ij^*} \partial_{j^*} \mathcal{P}_{(\mathbf{I})}$ is holomorphic, then eq.(2.40) defines a Killing vector.

The construction of the Killing prepotential can be stated in a more precise geometrical fashion through the notion of *moment map*. Let us review this construction.

Consider a Kählerian manifold \mathcal{M} of real dimension $2n$. Consider an isometry group \mathcal{G} acting on \mathcal{M} by means of Killing vector fields \vec{X} which are holomorphic with respect to the complex structure J of \mathcal{M} ; then these vector fields preserve also the Kähler 2-form

$$\left. \begin{array}{l} \mathcal{L}_{\vec{X}} g = 0 \\ \mathcal{L}_{\vec{X}} J = 0 \end{array} \right\} \Leftrightarrow \nabla_{(\mu} X_{\nu)} = 0 \quad \Rightarrow \quad 0 = \mathcal{L}_{\vec{X}} K = i_{\vec{X}} dK + d(i_{\vec{X}} K) = d(i_{\vec{X}} K) \quad (2.41)$$

Here $\mathcal{L}_{\vec{X}}$ and $i_{\vec{X}}$ denote respectively the Lie derivative along the vector field \vec{X} and the contraction (of forms) with it.

If \mathcal{M} is simply connected, $d(i_{\vec{X}} K) = 0$ implies the existence of a function $\mathcal{P}_{\vec{X}}$ such that

$$-\frac{1}{2} d\mathcal{P}_{\vec{X}} = i_{\vec{X}} K \quad (2.42)$$

The function $\mathcal{P}_{\vec{X}}$ is defined up to a constant, which can be arranged so as to make it equivariant:

$$\vec{X} \mathcal{P}_{\vec{Y}} = \mathcal{P}_{[\vec{X}, \vec{Y}]} \quad (2.43)$$

$\mathcal{P}_{\vec{X}}$ constitutes then a *moment map*. This can be regarded as a map

$$\mathcal{P} : \mathcal{M} \longrightarrow \mathbb{R} \otimes \mathbb{G}^* \quad (2.44)$$

where \mathbb{G}^* denotes the dual of the Lie algebra \mathbb{G} of the group \mathcal{G} . Indeed let $x \in \mathbb{G}$ be the Lie algebra element corresponding to the Killing vector \vec{X} ; then, for a given $m \in \mathcal{M}$

$$\mu(m) : x \longrightarrow \mathcal{P}_{\vec{X}}(m) \in \mathbb{R} \quad (2.45)$$

is a linear functional on \mathbb{G} . If we expand $\vec{X} = a^{\mathbf{I}} k_{\mathbf{I}}$ in a basis of Killing vectors $k_{\mathbf{I}}$ such that

$$[k_{\mathbf{I}}, k_{\mathbf{L}}] = f_{\mathbf{IL}}^{\mathbf{K}} k_{\mathbf{K}} \quad (2.46)$$

we have also

$$\mathcal{P}_{\vec{X}} = a^{\mathbf{I}} \mathcal{P}_{\mathbf{I}} \quad (2.47)$$

In the following we use the shorthand notation $\mathcal{L}_{\mathbf{I}}, i_{\mathbf{I}}$ for the Lie derivative and the contraction along the chosen basis of Killing vectors $k_{\mathbf{I}}$.

From a geometrical point of view the prepotential, or moment map, $\mathcal{P}_{\mathbf{I}}$ is the Hamiltonian function providing the Poissonian realization of the Lie algebra on the Kähler manifold. This is just another way of stating the already mentioned *equivariance*. Indeed the very existence of the closed 2-form K guarantees that every Kähler space is a symplectic manifold and that we can define a Poisson bracket.

Consider eqs.(2.40). To every generator of the abstract Lie algebra \mathbb{G} we have associated a function $\mathcal{P}_{\mathbf{I}}$ on \mathcal{M} ; the Poisson bracket of $\mathcal{P}_{\mathbf{I}}$ with $\mathcal{P}_{\mathbf{J}}$ is defined as follows:

$$\{\mathcal{P}_{\mathbf{I}}, \mathcal{P}_{\mathbf{J}}\} \equiv 4\pi K(\mathbf{I}, \mathbf{J}) \quad (2.48)$$

where $K(\mathbf{I}, \mathbf{J}) \equiv K(\vec{k}_{\mathbf{I}}, \vec{k}_{\mathbf{J}})$ is the value of K along the pair of Killing vectors.

In reference [55] the following lemma was proved:

Lemma 2.1 *The following identity is true:*

$$\{\mathcal{P}_{\mathbf{I}}, \mathcal{P}_{\mathbf{J}}\} = f_{\mathbf{IJ}}^{\mathbf{L}} \mathcal{P}_{\mathbf{L}} + C_{\mathbf{IJ}} \quad (2.49)$$

where $C_{\mathbf{IJ}}$ is a constant fulfilling the cocycle condition

$$f_{\mathbf{IM}}^{\mathbf{L}} C_{\mathbf{LJ}} + f_{\mathbf{MJ}}^{\mathbf{L}} C_{\mathbf{LI}} + f_{\mathbf{JI}}^{\mathbf{L}} C_{\mathbf{LM}} = 0 \quad (2.50)$$

If the Lie algebra \mathbb{G} has a trivial second cohomology group $H^2(\mathbb{G}) = 0$, then the cocycle $C_{\mathbf{IJ}}$ is a coboundary; namely we have

$$C_{\mathbf{IJ}} = f_{\mathbf{IJ}}^{\mathbf{L}} C_{\mathbf{L}} \quad (2.51)$$

where $C_{\mathbf{L}}$ are suitable constants. Hence, assuming $H^2(\mathbb{G}) = 0$ we can reabsorb $C_{\mathbf{L}}$ in the definition of $\mathcal{P}_{\mathbf{I}}$:

$$\mathcal{P}_{\mathbf{I}} \rightarrow \mathcal{P}_{\mathbf{I}} + C_{\mathbf{I}} \quad (2.52)$$

and we obtain the stronger equation

$$\{\mathcal{P}_{\mathbf{I}}, \mathcal{P}_{\mathbf{J}}\} = f_{\mathbf{IJ}}^{\mathbf{L}} \mathcal{P}_{\mathbf{L}} \quad (2.53)$$

Note that $H^2(\mathbb{G}) = 0$ is true for all semi-simple Lie algebras. Using eq.(2.49), eq.(2.53) can be rewritten in components as follows:

$$\frac{i}{2} g_{ij^*} (k_{\mathbf{I}}^i k_{\mathbf{J}}^{j^*} - k_{\mathbf{J}}^i k_{\mathbf{I}}^{j^*}) = \frac{1}{2} f_{\mathbf{IJ}}^{\mathbf{L}} \mathcal{P}_{\mathbf{L}} \quad (2.54)$$

Equation (2.54) is identical with the equivariance condition in eq.(2.43).

Finally let us recall the explicit general way of solving eq.(2.42) obtaining the real valued function $\mathcal{P}_{\mathbf{I}}$ which satisfies eq.(2.40). In terms of the Kähler potential \mathcal{K} we have:

$$\mathcal{P}_{\mathbf{I}}^x = -\frac{i}{2} (k_{\mathbf{I}}^i \partial_i \mathcal{K} - k_{\mathbf{I}}^{\bar{i}} \partial_{\bar{i}} \mathcal{K}) + \text{Im}(f_{\mathbf{I}}), \quad (2.55)$$

where $f_{\mathbf{I}} = f_{\mathbf{I}}(z)$ is a holomorphic transformation on the line-bundle, defining a compensating Kähler transformation:

$$k_{\mathbf{I}}^i \partial_i \mathcal{K} + k_{\mathbf{I}}^{\bar{i}} \partial_{\bar{i}} \mathcal{K} = -f_{\mathbf{I}}(z) - \bar{f}_{\mathbf{I}}(\bar{z}). \quad (2.56)$$

We also have:

$$\mathfrak{T}_{\mathbf{I}} \cdot \Omega = \mathfrak{T}_{\mathbf{I}} \cdot \Omega + f_{\mathbf{I}} \Omega, \quad (2.57)$$

$$\mathfrak{T}_{\mathbf{I}} \cdot V + i \operatorname{Im}(f_{\mathbf{I}}) V = k_{\mathbf{I}}^i \partial_i V + k_{\mathbf{I}}^{\bar{i}} \bar{\partial}_{\bar{i}} V, \quad (2.58)$$

where $\mathfrak{T}_{\mathbf{I}} \cdot \Omega$ denotes the symplectic action of the isometry on the section V . If $\mathfrak{T}_{\mathbf{I}}$ is represented by the symplectic matrix $(\mathfrak{T}_{\mathbf{I}})_{\alpha}^{\beta} = -(\mathfrak{T}_{\mathbf{I}})^{\beta}_{\alpha}$, $\alpha, \beta = 1, \dots, 2n+2$:

$$\mathfrak{T}_{\mathbf{I}}^T \mathbb{C} + \mathbb{C} \mathfrak{T}_{\mathbf{I}} = 0 \quad (2.59)$$

we have $(\mathfrak{T}_{\mathbf{I}} \cdot V)^{\alpha} = -\mathfrak{T}_{\mathbf{I}\beta}^{\alpha} V^{\beta} = \mathfrak{T}_{\mathbf{I}\beta}^{\alpha} V^{\beta}$. From (2.58) and (2.55) we derive the following useful symplectic-invariant expression for the moment maps:

$$\mathcal{P}_{\mathbf{I}}^x = -\bar{V}^{\alpha} \mathfrak{T}_{\mathbf{I}\alpha}^{\beta} \mathbb{C}_{\beta\gamma} V^{\gamma}. \quad (2.60)$$

Eq.s (2.55), (2.56), (2.58) generalize the corresponding formulae given in sections 7.1 and 7.2 of [63], where the condition $f_{\mathbf{I}} = 0$ was imposed, to gaugings of non-compact isometries which are associated with non-trivial compensating Kähler transformations and/or to gauged (non-compact) isometries whose symplectic action is not diagonal.

3 QUATERNIONIC GEOMETRY

Next we turn our attention to the geometry that pertains to the hypermultiplet sector of an $\mathcal{N} = 2$ supersymmetric theory. Each hypermultiplet contains 4 real scalar fields and, at least locally, they can be regarded as the four components of a quaternion. The locality caveat is, in this case, very substantial because global quaternionic coordinates can be constructed only occasionally even on those manifolds that are denominated quaternionic in the mathematical literature [64], [65]. Anyhow, what is important is that, in the hypermultiplet sector, the scalar manifold \mathcal{QM} has dimension multiple of four:

$$\dim_{\mathbf{R}} \mathcal{QM} = 4m \equiv 4 \# \text{ of hypermultiplets} \quad (3.1)$$

and, in some appropriate sense, it has a quaternionic structure.

We name *Hypergeometry* that pertaining to the hypermultiplet sector, irrespectively whether we deal with global or local $\mathcal{N}=2$ theories. Yet there are two kinds of hypergeometries. Supersymmetry requires the existence of a principal $SU(2)$ -bundle

$$SU \longrightarrow \mathcal{QM} \quad (3.2)$$

The bundle SU is **flat** in the *rigid supersymmetry case* while its curvature is proportional to the Kähler forms in the *local case*.

These two versions of hypergeometry were already known in mathematics prior to their use [55],[58], [61], [63], [64], [65] in the context of $\mathcal{N} = 2$ supersymmetry and are identified as:

$$\begin{aligned} \text{rigid hypergeometry} &\equiv \text{HyperKähler geometry.} \\ \text{local hypergeometry} &\equiv \text{Quaternionic Kähler geometry} \end{aligned} \quad (3.3)$$

3.1 QUATERNIONIC KÄHLER, VERSUS HYPERKÄHLER MANIFOLDS

Both a Quaternionic Kähler or a HyperKähler manifold \mathcal{QM} is a $4m$ -dimensional real manifold endowed with a metric h :

$$ds^2 = h_{uv}(q)dq^u \otimes dq^v \quad ; \quad u, v = 1, \dots, 4m \quad (3.4)$$

and three complex structures

$$(J^x) : T(\mathcal{QM}) \longrightarrow T(\mathcal{QM}) \quad (x = 1, 2, 3) \quad (3.5)$$

that satisfy the quaternionic algebra

$$J^x J^y = -\delta^{xy} \mathbb{1} + \epsilon^{xyz} J^z \quad (3.6)$$

and respect to which the metric is hermitian:

$$\forall \mathbf{X}, \mathbf{Y} \in T\mathcal{QM} : \quad h(J^x \mathbf{X}, J^x \mathbf{Y}) = h(\mathbf{X}, \mathbf{Y}) \quad (x = 1, 2, 3) \quad (3.7)$$

From eq. (3.7) it follows that one can introduce a triplet of 2-forms

$$K^x = K_{uv}^x dq^u \wedge dq^v \quad ; \quad K_{uv}^x = h_{uv}(J^x)_v^w \quad (3.8)$$

that provides the generalization of the concept of Kähler form occurring in the complex case. The triplet K^x is named the *HyperKähler* form. It is an $SU(2)$ Lie-algebra valued 2-form in the same way as the Kähler form is a $U(1)$ Lie-algebra valued 2-form. In the complex case the definition of Kähler manifold involves the statement that the Kähler 2-form is closed. At the same time in Hodge-Kähler manifolds the Kähler 2-form can be identified with the curvature of a line-bundle which in the case of rigid supersymmetry is flat. Similar steps can be taken also here and lead to two possibilities: either HyperKähler or Quaternionic Kähler manifolds.

Let us introduce a principal $SU(2)$ -bundle \mathcal{SU} as defined in eq. (3.2). Let ω^x denote a connection on such a bundle. To obtain either a HyperKähler or a Quaternionic Kähler manifold we must impose the condition that the HyperKähler 2-form is covariantly closed with respect to the connection ω^x :

$$\nabla K^x \equiv dK^x + \epsilon^{xyz} \omega^y \wedge K^z = 0 \quad (3.9)$$

The only difference between the two kinds of geometries resides in the structure of the \mathcal{SU} -bundle.

Definition 3.1 *A HyperKähler manifold is a $4m$ -dimensional manifold with the structure described above and such that the \mathcal{SU} -bundle is flat*

Defining the \mathcal{SU} -curvature by:

$$\Omega^x \equiv d\omega^x + \frac{1}{2} \epsilon^{xyz} \omega^y \wedge \omega^z \quad (3.10)$$

in the HyperKähler case we have:

$$\Omega^x = 0 \quad (3.11)$$

Viceversa

Definition 3.2 *A Quaternionic Kähler manifold is a $4m$ -dimensional manifold with the structure described above and such that the curvature of the \mathcal{SU} -bundle is proportional to the HyperKähler 2-form*

Hence, in the quaternionic case we can write:

$$\Omega^x = \lambda K^x \quad (3.12)$$

where λ is a non vanishing real number.

As a consequence of the above structure the manifold \mathcal{QM} has a holonomy group of the following type:

$$\begin{aligned} \text{Hol}(\mathcal{QM}) &= \text{SU}(2) \otimes \mathbb{H} \quad (\text{Quaternionic Kähler}) \\ \text{Hol}(\mathcal{QM}) &= \mathbb{1} \otimes \mathbb{H} \quad (\text{HyperKähler}) \\ \mathbb{H} &\subset \text{Sp}(2m, \mathbb{R}) \end{aligned} \quad (3.13)$$

In both cases, introducing flat indices $\{A, B, C = 1, 2\}\{\alpha, \beta, \gamma = 1, \dots, 2m\}$ that run, respectively, in the fundamental representation of $\text{SU}(2)$ and of $\text{Sp}(2m, \mathbb{R})$, we can find a vielbein 1-form

$$\mathcal{U}^{A\alpha} = \mathcal{U}_u^{A\alpha}(q) dq^u \quad (3.14)$$

such that

$$h_{uv} = \mathcal{U}_u^{A\alpha} \mathcal{U}_v^{B\beta} \mathbb{C}_{\alpha\beta} \epsilon_{AB} \quad (3.15)$$

where $\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}$ and $\epsilon_{AB} = -\epsilon_{BA}$ are, respectively, the flat $\text{Sp}(2m)$ and $\text{Sp}(2) \sim \text{SU}(2)$ invariant metrics. The vielbein $\mathcal{U}^{A\alpha}$ is covariantly closed with respect to the $\text{SU}(2)$ -connection ω^z and to some $\text{Sp}(2m, \mathbb{R})$ -Lie Algebra valued connection $\Delta^{\alpha\beta} = \Delta^{\beta\alpha}$:

$$\begin{aligned} \nabla \mathcal{U}^{A\alpha} &\equiv d\mathcal{U}^{A\alpha} + \frac{i}{2} \omega^x (\epsilon \sigma_x \epsilon^{-1})^A_B \wedge \mathcal{U}^{B\alpha} \\ &+ \Delta^{\alpha\beta} \wedge \mathcal{U}^{A\gamma} \mathbb{C}_{\beta\gamma} = 0 \end{aligned} \quad (3.16)$$

where $(\sigma^x)_A^B$ are the standard Pauli matrices. Furthermore $\mathcal{U}^{A\alpha}$ satisfies the reality condition:

$$\mathcal{U}_{A\alpha} \equiv (\mathcal{U}^{A\alpha})^* = \epsilon_{AB} \mathbb{C}_{\alpha\beta} \mathcal{U}^{B\beta} \quad (3.17)$$

Eq.(3.17) defines the rule to lower the symplectic indices by means of the flat symplectic metrics ϵ_{AB} and $\mathbb{C}_{\alpha\beta}$. More specifically we can write a stronger version of eq. (3.15) [62]:

$$(\mathcal{U}_u^{A\alpha} \mathcal{U}_v^{B\beta} + \mathcal{U}_v^{A\alpha} \mathcal{U}_u^{B\beta}) \mathbb{C}_{\alpha\beta} = h_{uv} \epsilon^{AB} \quad (3.18)$$

We have also the inverse vielbein $\mathcal{U}_{A\alpha}^u$ defined by the equation

$$\mathcal{U}_{A\alpha}^u \mathcal{U}_v^{A\alpha} = \delta_v^u \quad (3.19)$$

Flattening a pair of indices of the Riemann tensor \mathcal{R}^{uv}_{ts} we obtain

$$\mathcal{R}^{uv}_{ts} \mathcal{U}_u^{\alpha A} \mathcal{U}_v^{\beta B} = -\frac{i}{2} \Omega_{ts}^x \epsilon^{AC} (\sigma_x)_C^B \mathbb{C}^{\alpha\beta} + \mathbb{R}_{ts}^{\alpha\beta} \epsilon^{AB} \quad (3.20)$$

where $\mathbb{R}_{ts}^{\alpha\beta}$ is the field strength of the $\text{Sp}(2m)$ connection:

$$d\Delta^{\alpha\beta} + \Delta^{\alpha\gamma} \wedge \Delta^{\delta\beta} \mathbb{C}_{\gamma\delta} \equiv \mathbb{R}^{\alpha\beta} = \mathbb{R}_{ts}^{\alpha\beta} dq^t \wedge dq^s \quad (3.21)$$

Eq. (3.20) is the explicit statement that the Levi Civita connection associated with the metric h has a holonomy group contained in $SU(2) \otimes Sp(2m)$. Consider now eq.s (3.6), (3.8) and (3.12). We easily deduce the following relation:

$$h^{st} K_{us}^x K_{tw}^y = -\delta^{xy} h_{uw} + \epsilon^{xyz} K_{uw}^z \quad (3.22)$$

that holds true both in the HyperKähler and in the quaternionic case. In the latter case, using eq. (3.12), eq. (3.22) can be rewritten as follows:

$$h^{st} \Omega_{us}^x \Omega_{tw}^y = -\lambda^2 \delta^{xy} h_{uw} + \lambda \epsilon^{xyz} \Omega_{uw}^z \quad (3.23)$$

Eq.(3.23) implies that the intrinsic components of the curvature 2-form Ω^x yield a representation of the quaternion algebra. In the HyperKähler case such a representation is provided only by the HyperKähler form. In the quaternionic case we can write:

$$\Omega_{A\alpha, B\beta}^x \equiv \Omega_{uv}^x \mathcal{U}_{A\alpha}^u \mathcal{U}_{B\beta}^v = -i\lambda \mathbb{C}_{\alpha\beta}(\sigma_x)_A{}^C \epsilon_{CB} \quad (3.24)$$

Alternatively eq.(3.24) can be rewritten in an intrinsic form as

$$\Omega^x = -i\lambda \mathbb{C}_{\alpha\beta}(\sigma_x)_A{}^C \epsilon_{CB} \mathcal{U}^{\alpha A} \wedge \mathcal{U}^{\beta B} \quad (3.25)$$

whence we also get:

$$\frac{i}{2} \Omega^x(\sigma_x)_A{}^B = \lambda \mathcal{U}_{A\alpha} \wedge \mathcal{U}^{B\alpha} \quad (3.26)$$

3.2 THE TRIHOLOMORPHIC MOMENT MAP ON QUATERNIONIC MANIFOLDS

Next, following closely the original derivation of [55, 57] let us turn to a discussion of the triholomorphic isometries of the manifold \mathcal{QM} associated with hypermultiplets. In $D = 4$ supergravity the manifold of hypermultiplet scalars \mathcal{QM} is a Quaternionic Kähler manifold and we can gauge only those of its isometries that are triholomorphic and that either generate an abelian group \mathcal{G} or are *suitably realized* as isometries also on the special manifold $\widehat{\mathcal{SK}}_n$. This means that on \mathcal{QM} we have Killing vectors:

$$\vec{k}_{\mathbf{I}} = k_{\mathbf{I}}^u \frac{\vec{\partial}}{\partial q^u} \quad (3.27)$$

satisfying the same Lie algebra as the corresponding Killing vectors on $\widehat{\mathcal{SK}}_n$. In other words

$$\vec{\mathcal{K}}_{\mathbf{I}} = \hat{k}_{\mathbf{I}}^i \vec{\partial}_i + \hat{k}_{\mathbf{I}}^{i*} \vec{\partial}_{i^*} + k_{\mathbf{I}}^u \vec{\partial}_u \quad (3.28)$$

is a Killing vector of the block diagonal metric:

$$\mathfrak{g} = \begin{pmatrix} \hat{g}_{ij^*} & 0 \\ 0 & h_{uv} \end{pmatrix} \quad (3.29)$$

defined on the product manifold³ $\widehat{\mathcal{SK}} \otimes \mathcal{QM}$.

³Following the notations described in the introduction, the Special Kähler manifold which describes the interaction of vector multiplets is denoted $\widehat{\mathcal{SK}}$ and all the Special Geometry Structures are endowed with a hat in order to distinguish this Special Kähler manifold from the other one which is encapsulated into the Quaternionic Kähler manifold \mathcal{QM} describing the hypermultiplets when this latter happens to be in the image of the c -map.

Let us first focus on the manifold \mathcal{QM} . Triholomorphicity means that the Killing vector fields leave the HyperKähler structure invariant up to $SU(2)$ rotations in the $SU(2)$ -bundle defined by eq.(3.2). Namely:

$$\mathcal{L}_{\mathbf{I}}K^x = \epsilon^{xyz}K^yW_{\mathbf{I}}^z \quad ; \quad \mathcal{L}_{\mathbf{I}}\omega^x = \nabla W_{\mathbf{I}}^x \quad (3.30)$$

where $W_{\mathbf{I}}^x$ is an $SU(2)$ compensator associated with the Killing vector $k_{\mathbf{I}}^u$. The compensator $W_{\mathbf{I}}^x$ necessarily fulfills the cocycle condition:

$$\mathcal{L}_{\mathbf{I}}W_{\mathbf{J}}^x - \mathcal{L}_{\mathbf{J}}W_{\mathbf{I}}^x + \epsilon^{xyz}W_{\mathbf{I}}^yW_{\mathbf{J}}^z = f_{\mathbf{IJ}}^{\mathbf{L}}W_{\mathbf{L}}^x \quad (3.31)$$

In the HyperKähler case the $SU(2)$ -bundle is flat and the compensator can be reabsorbed into the definition of the HyperKähler forms. In other words we can always find a map

$$\mathcal{QM} \longrightarrow L^x_y(q) \in SO(3) \quad (3.32)$$

that trivializes the \mathcal{SU} -bundle globally. Redefining:

$$K^{x'} = L^x_y(q) K^y \quad (3.33)$$

the new HyperKähler form obeys the stronger equation:

$$\mathcal{L}_{\mathbf{I}}K^{x'} = 0 \quad (3.34)$$

On the other hand, in the quaternionic case, the non-triviality of the \mathcal{SU} -bundle forbids to eliminate the W -compensator completely. Due to the identification between HyperKähler forms and $SU(2)$ curvatures eq.(3.30) is rewritten as:

$$\mathcal{L}_{\mathbf{I}}\Omega^x = \epsilon^{xyz}\Omega^yW_{\mathbf{I}}^z \quad ; \quad \mathcal{L}_{\mathbf{I}}\omega^x = \nabla W_{\mathbf{I}}^x \quad (3.35)$$

In both cases, anyhow, and in full analogy with the case of Kähler manifolds, to each Killing vector we can associate a triplet $\mathcal{P}_{\mathbf{I}}^x(q)$ of 0-form prepotentials. Indeed we can set:

$$\mathbf{i}_{\mathbf{I}}K^x = -\nabla\mathcal{P}_{\mathbf{I}}^x \equiv -(d\mathcal{P}_{\mathbf{I}}^x + \epsilon^{xyz}\omega^y\mathcal{P}_{\mathbf{I}}^z) \quad (3.36)$$

where ∇ denotes the $SU(2)$ covariant exterior derivative.

As in the Kähler case eq.(3.36) defines a moment map:

$$\mathcal{P} : \mathcal{M} \longrightarrow \mathbb{R}^3 \otimes \mathcal{G}^* \quad (3.37)$$

where \mathcal{G}^* denotes the dual of the Lie algebra \mathcal{G} of the group \mathcal{G} . Indeed let $x \in \mathcal{G}$ be the Lie algebra element corresponding to the Killing vector \vec{X} ; then, for a given $m \in \mathcal{M}$

$$\mu(m) : x \longrightarrow \mathcal{P}_{\vec{X}}(m) \in \mathbb{R}^3 \quad (3.38)$$

is a linear functional on \mathcal{G} . If we expand $\vec{X} = a^{\mathbf{I}}k_{\mathbf{I}}$ on a basis of Killing vectors $k_{\mathbf{I}}$ such that

$$[k_{\mathbf{I}}, k_{\mathbf{L}}] = f_{\mathbf{IL}}^{\mathbf{K}}k_{\mathbf{K}} \quad (3.39)$$

and we also choose a basis \mathbf{i}_x ($x = 1, 2, 3$) for \mathbb{R}^3 we get:

$$\mathcal{P}_{\vec{X}} = a^{\mathbf{I}}\mathcal{P}_{\mathbf{I}}^x \mathbf{i}_x \quad (3.40)$$

Furthermore we need a generalization of the equivariance defined by eq.(2.43)

$$\vec{X} \circ \mathcal{P}_{\vec{Y}} = \mathcal{P}_{[\vec{X}, \vec{Y}]} \quad (3.41)$$

In the HyperKähler case, the left-hand side of eq.(3.41) is defined as the usual action of a vector field on a 0-form:

$$\vec{X} \circ \mathcal{P}_{\vec{Y}} = \mathbf{i}_{\vec{X}} d\mathcal{P}_{\vec{Y}} = X^u \frac{\partial}{\partial q^u} \mathcal{P}_{\vec{Y}} \quad (3.42)$$

The equivariance condition implies that we can introduce a triholomorphic Poisson bracket defined as follows:

$$\{\mathcal{P}_{\mathbf{I}}, \mathcal{P}_{\mathbf{J}}\}^x \equiv 2K^x(\mathbf{I}, \mathbf{J}) \quad (3.43)$$

leading to the triholomorphic Poissonian realization of the Lie algebra:

$$\{\mathcal{P}_{\mathbf{I}}, \mathcal{P}_{\mathbf{J}}\}^x = f_{\mathbf{IJ}}^{\mathbf{K}} \mathcal{P}_{\mathbf{K}}^x \quad (3.44)$$

which in components reads:

$$K_{uv}^x k_{\mathbf{I}}^u k_{\mathbf{J}}^v = \frac{1}{2} f_{\mathbf{IJ}}^{\mathbf{K}} \mathcal{P}_{\mathbf{K}}^x \quad (3.45)$$

In the quaternionic case, instead, the left-hand side of eq.(3.41) is interpreted as follows:

$$\vec{X} \circ \mathcal{P}_{\vec{Y}} = \mathbf{i}_{\vec{X}} \nabla \mathcal{P}_{\vec{Y}} = X^u \nabla_u \mathcal{P}_{\vec{Y}} \quad (3.46)$$

where ∇ is the $SU(2)$ -covariant differential. Correspondingly, the triholomorphic Poisson bracket is defined as follows:

$$\{\mathcal{P}_{\mathbf{I}}, \mathcal{P}_{\mathbf{J}}\}^x \equiv 2K^x(\mathbf{I}, \mathbf{J}) - \lambda \varepsilon^{xyz} \mathcal{P}_{\mathbf{I}}^y \mathcal{P}_{\mathbf{J}}^z \quad (3.47)$$

and leads to the Poissonian realization of the Lie algebra

$$\{\mathcal{P}_{\mathbf{I}}, \mathcal{P}_{\mathbf{J}}\}^x = f_{\mathbf{IJ}}^{\mathbf{K}} \mathcal{P}_{\mathbf{K}}^x \quad (3.48)$$

which in components reads:

$$K_{uv}^x k_{\mathbf{I}}^u k_{\mathbf{J}}^v - \frac{\lambda}{2} \varepsilon^{xyz} \mathcal{P}_{\mathbf{I}}^y \mathcal{P}_{\mathbf{J}}^z = \frac{1}{2} f_{\mathbf{IJ}}^{\mathbf{K}} \mathcal{P}_{\mathbf{K}}^x \quad (3.49)$$

Eq.(3.49), which is the most convenient way of expressing equivariance in a coordinate basis was originally written in [55] and has played a fundamental role in the construction of supersymmetric actions for gauged $\mathcal{N} = 2$ supergravity both in $D = 4$ [55, 63] and in $D = 5$ [66].

4 THE QUATERNIONIC KÄHLER GEOMETRY IN THE IMAGE OF THE c -MAP

The main object of study in the present paper are those Quaternionic Kähler manifolds that are in the image of the c -map.⁴ This latter

$$\text{c-map} : \mathcal{SK}_n \Longrightarrow \mathcal{QM}_{4n+4} \quad (4.1)$$

is a universal construction that starting from an arbitrary Special Kähler manifold \mathcal{SK}_n of complex dimension n , irrespectively whether it is homogeneus or not, leads to a unique Quaternionic Kähler manifold \mathcal{QM}_{4n+4} of real dimension $4n + 4$ which contains \mathcal{SK}_n as a submanifold. The precise modern definition of the c -map, originally introduced in [2], is provided below.

⁴Not all non-compact, homogeneous Quaternionic Kähler manifolds which are relevant to supergravity (which are *normal*, i.e. exhibiting a solvable group of isometries having a free and transitive action on it) are in the image of the c -map, the only exception being the quaternionic projective spaces [59, 60, 61].

Definition 4.1 Let SK_n be a special Kähler manifold whose complex coordinates we denote by z^i and whose Kähler metric we denote by g_{ij}^* . Let moreover $\mathcal{N}_{\Lambda\Sigma}(z, \bar{z})$ be the symmetric period matrix defined by eq.(2.29), introduce the following set of $4n + 4$ coordinates:

$$\{q^u\} \equiv \underbrace{\{U, a\}}_{2 \text{ real}} \cup \underbrace{\{z^i\}}_{\substack{n \text{ complex} \\ 2n \text{ real}}} \cup \underbrace{\mathbf{Z} = \{Z^\Lambda, Z_\Sigma\}}_{(2n+2) \text{ real}} \quad (4.2)$$

Let us further introduce the following $(2n + 2) \times (2n + 2)$ matrix \mathcal{M}_4^{-1} :

$$\mathcal{M}_4^{-1} = \left(\begin{array}{c|c} \text{Im}\mathcal{N} + \text{Re}\mathcal{N} \text{Im}\mathcal{N}^{-1} \text{Re}\mathcal{N} & -\text{Re}\mathcal{N} \text{Im}\mathcal{N}^{-1} \\ \hline -\text{Im}\mathcal{N}^{-1} \text{Re}\mathcal{N} & \text{Im}\mathcal{N}^{-1} \end{array} \right) \quad (4.3)$$

which depends only on the coordinate of the Special Kähler manifold. The c -map image of SK_n is the unique Quaternionic Kähler manifold \mathcal{QM}_{4n+4} whose coordinates are the q^u defined in (4.2) and whose metric is given by the following universal formula

$$ds_{\mathcal{QM}}^2 = \frac{1}{4} \left(dU^2 + 4g_{ij}^* dz^i d\bar{z}^{j*} + e^{-2U} (da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z})^2 - 2e^{-U} d\mathbf{Z}^T \mathcal{M}_4^{-1} d\mathbf{Z} \right) \quad (4.4)$$

The metric (4.4) has the following positive definite signature

$$\text{sign} [ds_{\mathcal{QM}}^2] = \left(\underbrace{+, \dots, +}_{4+4n} \right) \quad (4.5)$$

since the matrix \mathcal{M}_4^{-1} is negative definite.

It is worth mentioning that if we utilize the same $4n + 4$ coordinates (4.2) and instead of (4.4) we introduce the alternative Lorentzian metric:

$$ds_{\mathcal{QM}^*}^2 = \frac{1}{4} \left(dU^2 + 4g_{ij}^* dz^i d\bar{z}^{j*} + e^{-2U} (da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z})^2 + 2e^{-U} d\mathbf{Z}^T \mathcal{M}_4 d\mathbf{Z} \right) \quad (4.6)$$

that has signature:

$$\text{sign} [ds_{\mathcal{QM}^*}^2] = \left(\underbrace{+, \dots, +}_{2n+2}, \underbrace{-, \dots, -}_{2n+2} \right) \quad (4.7)$$

we obtain the pseudo-quaternionic manifold \mathcal{QM} which constitutes the target manifold in the 3-dimensional σ -model description of $D = 4$ supergravity Black-Hole solutions [71],[4],[70]. In the case the Special Kähler pre-image is a symmetric space U_{SK}/H_{SK} , both \mathcal{QM} and \mathcal{QM}^* turn out to be symmetric spaces as well, U_Q/H_Q and U_Q/H_Q^* , the numerator group being the same. We will come back to the issue of symmetric homogeneous Quaternionic Kähler manifolds in section 4.3

4.1 THE HYPERKÄHLER TWO-FORMS AND THE $\mathfrak{su}(2)$ -CONNECTION

The reason why we state that \mathcal{QM}_{4n+4} is Quaternionic Kähler is that, by utilizing only the identities of Special Kähler Geometry we can construct the three complex structures $J_u^{x|v}$ satisfying the quaternionic

algebra (3.6) the corresponding HyperKähler two-forms K^x and the $\mathfrak{su}(2)$ connection ω^x with respect to which they are covariantly constant.

The construction is extremely beautiful and it is the following one.

Consider the Kähler connection \mathcal{Q} defined by eq. (2.8) and furthermore introduce the following differential form:

$$\Phi = da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z} \quad (4.8)$$

Next define the two dimensional representation of both the $\mathfrak{su}(2)$ connection and of the HyperKähler 2-forms as it follows:

$$\omega = \frac{i}{\sqrt{2}} \sum_{x=1}^3 \omega^x \gamma_x \quad (4.9)$$

$$\mathbf{K} = \frac{i}{\sqrt{2}} \sum_{x=1}^3 K^x \sigma_x \quad (4.10)$$

where γ_x denotes a basis of 2×2 euclidian γ -matrices for which we utilize the following basis which is convenient in the explicit calculations we perform in Part Four⁵:

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \gamma_2 &= \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \\ \gamma_3 &= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \end{aligned} \quad (4.11)$$

These γ -matrices satisfy the following Clifford algebra:

$$\{\gamma_x, \gamma_y\} = \delta^{xy} \mathbf{1}_{2 \times 2} \quad (4.12)$$

and $\frac{i}{2} \gamma_x$ provide a basis of generators of the $\mathfrak{su}(2)$ algebra.

Having fixed these conventions the expression of the quaternionic $\mathfrak{su}(2)$ -connection in terms of Special Geometry structures is encoded in the following expression for the 2×2 -matrix valued 1-form ω . Explicitly we have:

$$\omega = \begin{pmatrix} -\frac{i}{2} \mathcal{Q} - \frac{i}{4} e^{-U} \Phi & e^{-\frac{U}{2}} V^T \mathbb{C} d\mathbf{Z} \\ -e^{-\frac{U}{2}} \bar{V}^T \mathbb{C} d\mathbf{Z} & \frac{i}{2} \mathcal{Q} + \frac{i}{4} e^{-U} \Phi \end{pmatrix} \quad (4.13)$$

where V and \bar{V} denote the covariantly holomorphic sections of Special geometry defined in eq.s (2.19). The curvature of this connection is obtained from a straight-forward calculation:

$$\begin{aligned} \mathbf{K} &\equiv d\omega + \omega \wedge \omega \\ &= \begin{pmatrix} \mathbf{u} & \mathbf{v} \\ -\bar{\mathbf{v}} & -\mathbf{u} \end{pmatrix} \end{aligned} \quad (4.14)$$

⁵The chosen γ -matrices are a permutation of the standard pauli matrices divided by $\sqrt{2}$ and multiplied by $\frac{i}{2}$ can be used as a basis of anti-hermitian generators for the $\mathfrak{su}(2)$ algebra in the fundamental defining representation.

the independent 2-form matrix elements being given by the following explicit formulae:

$$\begin{aligned}
\mathbf{u} &= -i\frac{1}{2}K - \frac{1}{8}dS \wedge d\bar{S} - e^{-U} V^T \mathbb{C} d\mathbf{Z} \wedge \bar{V}^T \mathbb{C} d\mathbf{Z} - \frac{1}{4}e^{-U} d\mathbf{Z}^T \wedge \mathbb{C} d\mathbf{Z} \\
\mathbf{v} &= e^{-\frac{U}{2}} \left(DV^T \wedge \mathbb{C} d\mathbf{Z} - \frac{1}{2}dS \wedge V^T \mathbb{C} d\mathbf{Z} \right) \\
\bar{\mathbf{v}} &= e^{-\frac{U}{2}} \left(D\bar{V}^T \wedge \mathbb{C} d\mathbf{Z} - \frac{1}{2}d\bar{S} \wedge \bar{V}^T \mathbb{C} d\mathbf{Z} \right)
\end{aligned} \tag{4.15}$$

where

$$K = \frac{i}{2} g_{ij^*} dz^i \wedge d\bar{z}^{j^*} \tag{4.16}$$

is the Kähler 2-form of the Special Kähler submanifold and where we have used the following short hand notations:

$$dS = dU + ie^{-U} (da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z}) \tag{4.17}$$

$$d\bar{S} = dU - ie^{-U} (da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z}) \tag{4.18}$$

$$DV = dz^i \nabla_i V \tag{4.19}$$

$$D\bar{V} = d\bar{z}^{i^*} \nabla_{i^*} V \tag{4.20}$$

The three HyperKähler forms K^x are easily extracted from eq.s (4.14-4.15) by collecting the coefficients of the γ -matrix expansion and we need not to write their form which is immediately deduced. The relevant thing is that the components of K^x with an index raised through multiplication with the inverse of the quaternionic metric h^{uv} exactly satisfy the algebra of quaternionic complex structures (3.6). Explicitly we have:

$$\begin{aligned}
K^x &= -i4\sqrt{2} \text{Tr} (\gamma^x \mathbf{K}) \equiv K_{uv}^x dq^u \wedge dq^v \\
J_u^{x|s} &= K_{uv}^x h^{vs} \\
J_u^{x|s} J_s^{y|v} &= -\delta^{xy} \delta_u^v + \epsilon^{xyz} J_u^{z|v}
\end{aligned} \tag{4.21}$$

The above formulae are not only the general proof that the Riemannian manifold \mathcal{QM} defined by the metric (4.4) is indeed a Quaternionic Kähler manifold, but, what is most relevant, they also provide an algorithm to write in terms of Special Geometry structures the tri-holomorphic moment map of the principal isometries possessed by \mathcal{QM} .

4.2 ISOMETRIES OF \mathcal{QM} IN THE IMAGE OF THE c -MAP AND THEIR TRI-HOLOMORPHIC MOMENT MAPS

Let us now consider the isometries of the metric (4.4). There are three type of isometries:

- a) The isometries of the $(2n+3)$ -dimensional Heisenberg algebra \mathbb{H}_{eis} which is always present and is universal for any $(4n+4)$ -dimensional Quaternionic Kähler manifold in the image of the c -map. We describe it below.
- b) All the isometries of the pre-image Special Kähler manifold \mathcal{SK}_n that are promoted to isometries of the image manifold in a way described below.
- c) The additional $2n+4$ isometries that occur only when \mathcal{SK}_n is a symmetric space and such, as a consequence, is also the c -map image \mathcal{QM}_{4n+4} . We will discuss these isometries in section (4.3).

For the first two types of isometries a) and b) we are able to write general expressions for the tri-holomorphic moment maps that utilize only the structures of Special Geometry. In the case that the additional isometries c) do exist we have another universal formula which can be used for all generators of the isometry algebra \mathbb{U}_Q and which relies on the identification of the generators of the $\mathfrak{su}(2) \subset \mathbb{H}$ subalgebra with the three complex structures. We will illustrate the details of such an identification while discussing the example of the S^3 -model.

First of all let us fix the notation writing the general form of a Killing vector. This a tangent vector:

$$\begin{aligned}\vec{\mathbf{k}} &= k^u(q) \partial_u \\ &= k^\diamond \frac{\partial}{\partial U} + k^i \frac{\partial}{\partial z^i} + k^{i*} \frac{\partial}{\partial \bar{z}^{i*}} + k^\bullet \frac{\partial}{\partial a} + k^\alpha \frac{\partial}{\partial \mathbf{Z}^\alpha} \\ &\equiv k^\diamond \partial_\diamond + k^i \partial_i + k^{i*} \partial_{i*} + k^\bullet \partial_\bullet + k^\alpha \partial_\alpha\end{aligned}\tag{4.22}$$

with respect to which the Lie derivative of the metric element (4.4) vanishes:

$$\ell_{\vec{\mathbf{k}}} ds_{\mathcal{QM}}^2 = 0\tag{4.23}$$

4.2.1 TRI-HOLOMORPHIC MOMENT MAPS FOR THE HEISENBERG ALGEBRA TRANSLATIONS

First let us consider the isometries associated with the Heisenberg algebra. The transformation:

$$Z^\alpha \mapsto Z^\alpha + \Lambda^\alpha \quad ; \quad a \mapsto a - \Lambda^T \mathbb{C} \mathbf{Z}\tag{4.24}$$

where Λ^α is an arbitrary set of $2n+2$ real infinitesimal parameters is an infinitesimal isometry for the metric $ds_{\mathcal{QM}}^2$ in (4.4). It corresponds to the following Killing vector:

$$\begin{aligned}\vec{\mathbf{k}}_{[\Lambda]} &= \Lambda^\alpha \vec{\mathbf{k}}_\alpha \\ &= \Lambda^\alpha \partial_\alpha - \Lambda^T \mathbb{C} \mathbf{Z} \partial_\bullet\end{aligned}\tag{4.25}$$

whose components are immediately deduced by comparison of eq.(4.25) with eq.(4.22).

We are interested in determining the expression of the tri-holomorphic moment map $\mathfrak{P}_{[\Lambda]}$ which satisfies the defining equation:

$$\mathbf{i}_{[\Lambda]} \mathbf{K} \equiv \begin{pmatrix} \mathbf{i}_{[\Lambda]} \mathbf{u} & \mathbf{i}_{[\Lambda]} \mathbf{v} \\ -\mathbf{i}_{[\Lambda]} \bar{\mathbf{v}} & -\mathbf{i}_{[\Lambda]} \mathbf{u} \end{pmatrix} = d\mathfrak{P}_{[\Lambda]} + [\omega, \mathfrak{P}_{[\Lambda]}\tag{4.26}$$

The general solution to this problem is

$$\mathfrak{P}_{[\Lambda]} = \begin{pmatrix} -\frac{i}{4} e^{-U} \Lambda^T \mathbb{C} \mathbf{Z} & \frac{1}{2} e^{-\frac{U}{2}} \Lambda^T C V \\ -\frac{1}{2} e^{-\frac{U}{2}} \Lambda^T C \bar{V} & \frac{i}{4} e^{-U} \Lambda^T \mathbb{C} \mathbf{Z} \end{pmatrix}\tag{4.27}$$

4.2.2 TRI-HOLOMORPHIC MOMENT MAP FOR THE HEISENBERG ALGEBRA CENTRAL CHARGE

Consider next the isometry associated with the Heisenberg algebra central charge. The transformation:

$$a \mapsto a + \varepsilon\tag{4.28}$$

where ε is an arbitrary real small parameter is an infinitesimal isometry for the metric $ds_{\mathcal{QM}}^2$ in (4.4). It corresponds to the following Killing vector:

$$\varepsilon \vec{\mathbf{k}}_{[\bullet]} = \varepsilon \partial_{\bullet} \quad (4.29)$$

whose components are immediately deduced by comparison of eq.(4.29) with eq.(4.22).

We are interested in determining the expression of the tri-holomorphic moment map $\mathfrak{P}_{[\bullet]}$ which satisfies the defining equation analogous to eq.(4.26):

$$\mathbf{i}_{[\bullet]} \mathbf{K} = d\mathfrak{P}_{[\bullet]} + [\omega, \mathfrak{P}_{[\bullet]}] \quad (4.30)$$

The solution of this problem is even simpler than in the previous case. Explicitly we obtain:

$$\mathfrak{P}_{[\bullet]} = \begin{pmatrix} -\frac{i}{8} e^{-U} & 0 \\ 0 & \frac{i}{8} e^{-U} \end{pmatrix} \quad (4.31)$$

The explicit expression of the moment maps and Killing vectors associated with the Heisenberg isometries was used in the gauging of abelian subalgebras of the Heisenberg algebra, which is relevant to the description of compactifications of Type II superstring on a generalized Calabi-Yau manifold [67].

4.2.3 TRI-HOLOMORPHIC MOMENT MAP FOR THE EXTENSION OF \mathcal{SK}_n HOLOMORPHIC ISOMETRIES

Next we consider the question how to write the moment map associated with those isometries that were already present in the original Special Kähler manifold \mathcal{SK}_n which we c -mapped to a Quaternionic Kähler manifold.

Suppose that \mathcal{SK}_n has a certain number of holomorphic Killing vectors $k_{\mathbf{I}}^i(z)$ satisfying eq.s (2.39,2.40,2.41) necessarily closing some Lie algebra $\mathfrak{g}_{\mathcal{SK}}$ among themselves. Their holomorphic momentum-map is provided by eq.(2.55). Necessarily every isometry of a special Kähler manifold has a linear symplectic $(2n+2)$ -dimensional realization on the holomorphic section $\Omega(z)$ up to an overall holomorphic factor. This means that for each holomorphic Killing vector we have (see Eq. (2.57)):

$$k_{\mathbf{I}}^i(z) \partial_i \Omega(z) = \exp[f_{\mathbf{I}}(z)] \mathfrak{T}_{\mathbf{I}} \Omega(z). \quad (4.32)$$

where $f_{\mathbf{I}}(z)$ the holomorphic Kähler compensator. Then it can be easily checked that the transformation:

$$z^i \mapsto z^i + k_{\mathbf{I}}^i(z) \quad ; \quad \mathbf{Z} \mapsto \mathbf{Z} + \mathfrak{T}_{\mathbf{I}} \mathbf{Z} \quad (4.33)$$

is an infinitesimal isometry of the metric (4.4) corresponding to the Killing vector:

$$\vec{\mathbf{k}}_{\mathbf{I}} = k_{\mathbf{I}}^i(z) \partial_i + k_{\mathbf{I}}^{i*}(\bar{z}) \partial_{i^*} + (\mathfrak{T}_{\mathbf{I}})_{\beta}^{\alpha} \mathbf{Z}^{\beta} \partial_{\alpha} \quad (4.34)$$

Also in this case we are interested in determining the expression of the tri-holomorphic moment map $\mathfrak{P}_{[\mathbf{I}]}$ satisfying the defining equation:

$$\mathbf{i}_{\vec{\mathbf{k}}_{\mathbf{I}}} \mathbf{K} = d\mathfrak{P}_{[\mathbf{I}]} + [\omega, \mathfrak{P}_{[\mathbf{I}]}] \quad (4.35)$$

The solution is given by the expression below:

$$\mathfrak{P}_{[\mathbf{I}]} = \begin{pmatrix} \frac{i}{4} (\mathcal{P}_{\mathbf{I}} + \frac{1}{2} e^{-U} \mathbf{Z}^T \mathbb{C} \mathfrak{T}_{\mathbf{I}} \mathbf{Z}) & -\frac{1}{2} e^{-U/2} V^T \mathbb{C} \mathfrak{T}_{\mathbf{I}} \mathbf{Z} \\ \frac{1}{2} e^{-U/2} \bar{V}^T \mathbb{C} \mathfrak{T}_{\mathbf{I}} \mathbf{Z} & -\frac{i}{4} (\mathcal{P}_{\mathbf{I}} + \frac{1}{2} e^{-U} \mathbf{Z}^T \mathbb{C} \mathfrak{T}_{\mathbf{I}} \mathbf{Z}) \end{pmatrix} \quad (4.36)$$

where $\mathcal{P}_{\mathbf{I}}$ is the moment map of the same Killing vector in pure Special Geometry.

4.3 HOMOGENEOUS SYMMETRIC SPECIAL QUATERNIONIC KÄHLER MANIFOLDS

When the Special Kähler manifold \mathcal{SK}_n is a symmetric coset space, it turns out that the metric (4.4) is actually the symmetric metric on an enlarged symmetric coset manifold

$$\mathcal{QM}_{4n+4} = \frac{\mathcal{U}_Q}{\mathcal{H}_Q} \supset \frac{\mathcal{U}_{\mathcal{SK}}}{\mathcal{H}_{\mathcal{SK}}} \quad (4.37)$$

Naming $\Lambda[\mathfrak{g}]$ the \mathbf{W} -representation of any finite element of the $\mathfrak{g} \in \mathcal{U}_{\mathcal{SK}}$ group, we have that the matrix $\mathcal{M}_4(z, \bar{z})$ transforms as follows:

$$\mathcal{M}_4(\mathfrak{g} \cdot z, \mathfrak{g} \cdot \bar{z}) = \Lambda[\mathfrak{g}] \mathcal{M}_4(z, \bar{z}) \Lambda^T[\mathfrak{g}] \quad (4.38)$$

where $\mathfrak{g} \cdot z$ denotes the non linear action of $\mathcal{U}_{\mathcal{SK}}$ on the scalar fields. Since the space $\frac{\mathcal{U}_{\mathcal{SK}}}{\mathcal{H}_{\mathcal{SK}}}$ is homogeneous, choosing any reference point z_0 all the others can be reached by a suitable group element \mathfrak{g}_z such that $\mathfrak{g}_z \cdot z_0 = z$ and we can write:

$$\mathcal{M}_4^{-1}(z, \bar{z}) = \Lambda^T[\mathfrak{g}_z^{-1}] \mathcal{M}_4^{-1}(z_0, \bar{z}_0) \Lambda[\mathfrak{g}_z^{-1}] \quad (4.39)$$

This allows to introduce a set of $4n + 4$ vielbein defined in the following way:

$$E_{\mathcal{QM}}^I = \frac{1}{2} \left\{ dU, \underbrace{e^i(z)}_{2n}, e^{-U} (da + \mathbf{Z}^T \mathbb{C} d\mathbf{Z}), \underbrace{e^{-\frac{U}{2}} \Lambda[\mathfrak{g}_z^{-1}] d\mathbf{Z}}_{2n+2} \right\} \quad (4.40)$$

and rewrite the metric (4.4) as it follows:

$$ds_{\mathcal{QM}}^2 = E_{\mathcal{QM}}^I \mathfrak{q}_{IJ} E_{\mathcal{QM}}^J \quad (4.41)$$

where the quadratic symmetric constant tensor \mathfrak{q}_{IJ} has the following form:

$$\mathfrak{q}_{IJ} = \left(\begin{array}{c|c|c|c} 1 & 0 & 0 & 0 \\ \hline 0 & \delta_{ij} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & -2 \mathcal{M}_4^{-1}(z_0, \bar{z}_0) \end{array} \right) \quad (4.42)$$

The above defined vielbein are endowed with a very special property namely they identically satisfy a set of Maurer Cartan equations:

$$dE_{\mathcal{QM}}^I - \frac{1}{2} f^I_{JK} E_{\mathcal{QM}}^J \wedge E_{\mathcal{QM}}^K = 0 \quad (4.43)$$

where f^I_{JK} are the structure constants of a solvable Lie algebra \mathfrak{A} which can be identified as follows:

$$\mathfrak{A} = \text{Solv} \left(\frac{\mathcal{U}_Q}{\mathcal{H}_Q} \right) \quad (4.44)$$

In the above equation $\text{Solv} \left(\frac{\mathcal{U}_Q}{\mathcal{H}_Q} \right)$ denotes the Lie algebra of the solvable group manifold metrically equivalent to the non-compact coset manifold $\frac{\mathcal{U}_Q}{\mathcal{H}_Q}$ according to a well developed mathematical theory extensively used

in supergravity theories[68]. In the case $\mathbb{U}_{\mathcal{K}}$ is a *maximally split* real form of a complex Lie algebra, then also $\mathbb{U}_{\mathcal{Q}}$ is maximally split and we have:

$$\text{Solv}\left(\frac{\mathbb{U}_{\mathcal{Q}}}{\mathbb{H}_{\mathcal{Q}}}\right) = \text{Bor}(\mathbb{U}_{\mathcal{Q}}) \quad (4.45)$$

where $\text{Bor}(\mathbb{U}_{\mathcal{Q}})$ denotes the *Borel subalgebra* of the semi-simple Lie algebra \mathbb{G} , generated by its Cartan generators and by the step operators associated with all positive roots.

According to the general mathematical theory mentioned above, the very fact that the vielbein (4.40) satisfies the Maurer-Cartan equations of $\text{Solv}\left(\frac{\mathbb{U}_{\mathcal{Q}}}{\mathbb{H}_{\mathcal{Q}}}\right)$ implies that the metric (4.41) is the symmetric metric on the coset manifold $\frac{\mathbb{U}_{\mathcal{Q}}}{\mathbb{H}_{\mathcal{Q}}}$ which therefore admits continuous isometries associated with all the generators of the Lie algebra $\mathbb{U}_{\mathcal{Q}}$. This latter admits the following general decomposition:

$$\text{adj}(\mathbb{U}_{\mathcal{Q}}) = \text{adj}(\mathbb{U}_{\mathcal{K}}) \oplus \text{adj}(\text{SL}(2, \mathbb{R})_{\text{E}}) \oplus \mathbf{W}_{(2, \mathbf{W})} \quad (4.46)$$

where \mathbf{W} is the **symplectic** representation of $\mathbb{U}_{\mathcal{K}}$ in which the symplectic section of Special Geometry transforms and which was used to construct the vielbein (4.41). Denoting the generators of $\mathbb{U}_{\mathcal{K}}$ by T^a , the generators of $\text{SL}(2, \mathbb{R})_{\text{E}}$ by L^x and denoting by $\mathbf{W}^{i\alpha}$ the generators in $\mathbf{W}_{(2, \mathbf{W})}$, the commutation relations that correspond to the decomposition (4.46) have the following general form [69]:

$$\begin{aligned} [T^a, T^b] &= f^{ab}_c T^c \\ [L^x_E, L^y_E] &= f^{xy}_z L^z, \\ [T^a, \mathbf{W}^{i\alpha}] &= (\Lambda^a)^\alpha_\beta \mathbf{W}^{i\beta}, \\ [L^x_E, \mathbf{W}^{i\alpha}] &= (\lambda^x)^i_j \mathbf{W}^{j\alpha}, \\ [\mathbf{W}^{i\alpha}, \mathbf{W}^{j\beta}] &= \epsilon^{ij} (K_a)^{\alpha\beta} T^a + \mathbb{C}^{\alpha\beta} k^{ij}_x L^x_E \end{aligned} \quad (4.47)$$

where the 2×2 matrices $(\lambda^x)^i_j$, are the canonical generators of $\text{SL}(2, \mathbb{R})$ in the fundamental, defining representation:

$$\lambda^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \quad ; \quad \lambda^1 = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \quad ; \quad \lambda^2 = \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix} \quad (4.48)$$

while Λ^a are the generators of $\mathbb{U}_{\mathcal{K}}$ in the symplectic representation \mathbf{W} . By

$$\mathbb{C}^{\alpha\beta} \equiv \left(\begin{array}{c|c} \mathbf{0}_{(n+1) \times (n+1)} & \mathbf{1}_{(n+1) \times (n+1)} \\ \hline -\mathbf{1}_{(n+1) \times (n+1)} & \mathbf{0}_{(n+1) \times (n+1)} \end{array} \right) \quad (4.49)$$

we denote the antisymmetric symplectic metric in $2n + 2$ dimensions, n being the complex dimension of the Special Kähler manifold $\frac{\mathbb{U}_{\mathcal{K}}}{\mathbb{H}_{\mathcal{K}}}$. The symplectic character of the representation \mathbf{W} is asserted by the identity:

$$\Lambda^a \mathbb{C} + \mathbb{C} (\Lambda^a)^T = 0 \quad (4.50)$$

The fundamental doublet representation of $\text{SL}(2, \mathbb{R})_{\text{E}}$ is also symplectic and we have denoted by $\epsilon^{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the 2-dimensional symplectic metric, so that:

$$\lambda^x \epsilon + \epsilon (\lambda^x)^T = 0, \quad (4.51)$$

The matrices $(K_a)^{\alpha\beta} = (K_a)^{\beta\alpha}$ and $(k_x)^{ij} = (k_y)^{ji}$ are just symmetric matrices in one-to-one correspondence with the generators of \mathbb{U}_Q and $\text{SL}(2, \mathbb{R})$, respectively. Implementing Jacobi identities we find the following relations:

$$K_a \Lambda^c + \Lambda^c K_a = f^{bc}_a K_b, \quad k_x \lambda^y + \lambda^y k_x = f^{yz}_x k_z,$$

which admit the unique solution:

$$K_a = c_1 \mathbf{g}_{ab} \Lambda^b \mathbb{C}, \quad ; \quad k_x = c_2 \mathbf{g}_{xy} \lambda^y \epsilon \quad (4.52)$$

where \mathbf{g}_{ab} , \mathbf{g}_{xy} are the Cartan-Killing metrics on the algebras $\mathbb{U}_{\mathcal{SK}}$ and $\text{SL}(2, \mathbb{R})$, respectively and c_1 and c_2 are two arbitrary constants. These latter can always be reabsorbed into the normalization of the generators $\mathbf{W}^{i\alpha}$ and correspondingly set to one. Hence the algebra (4.47) can always be put into the following elegant form:

$$\begin{aligned} [T^a, T^b] &= f^{ab}_c T^c \\ [L^x, L^y] &= f^{xy}_z L^z, \\ [T^a, \mathbf{W}^{i\alpha}] &= (\Lambda^a)^\alpha_\beta \mathbf{W}^{i\beta}, \\ [L^x, \mathbf{W}^{i\alpha}] &= (\lambda^x)^i_j \mathbf{W}^{j\alpha}, \\ [\mathbf{W}^{i\alpha}, \mathbf{W}^{j\beta}] &= \epsilon^{ij} (\Lambda_a)^{\alpha\beta} T^a + \mathbb{C}^{\alpha\beta} \lambda^{ij}_x L^x \end{aligned} \quad (4.53)$$

where we have used the convention that symplectic indices are raised and lowered with the symplectic metric, while adjoint representation indices are raised and lowered with the Cartan-Killing metric.

For the reader's convenience the list of Symmetric Special manifolds and of their Quaternionic Kähler counterparts in the image of the c -map is recalled in table 1 which reproduces the results of [56], according to which there is a short list of Symmetric Homogeneous Special manifolds comprising five discrete cases and two infinite series.

Inspecting eq.s (4.53) we immediately realize that the Lie Algebra \mathbb{U}_Q contains two universal Heisenberg subalgebras of dimension $(2n + 3)$, namely:

$$\begin{aligned} \mathbb{U}_Q \supset \text{Heis}_1 &= \text{span}_{\mathbb{R}} \{ \mathbf{W}^{1\alpha}, \mathbb{Z}_1 \} \quad ; \quad \mathbb{Z}_1 = L_+ \equiv L^1 + L^2 \\ [\mathbf{W}^{1\alpha}, \mathbf{W}^{1\beta}] &= -\frac{1}{2} \mathbb{C}^{\alpha\beta} \mathbb{Z}_1 \quad ; \quad [\mathbb{Z}_1, \mathbf{W}^{1\beta}] = 0 \end{aligned} \quad (4.54)$$

$$\begin{aligned} \mathbb{U}_Q \supset \text{Heis}_2 &= \text{span}_{\mathbb{R}} \{ \mathbf{W}^{2\alpha}, \mathbb{Z}_2 \} \quad ; \quad \mathbb{Z}_2 = L_- \equiv L^1 - L^2 \\ [\mathbf{W}^{2\alpha}, \mathbf{W}^{2\beta}] &= -\frac{1}{2} \mathbb{C}^{\alpha\beta} \mathbb{Z}_2 \quad ; \quad [\mathbb{Z}_2, \mathbf{W}^{2\beta}] = 0 \end{aligned} \quad (4.55)$$

The first of these Heisenberg subalgebras of isometries is the universal one that exists for all Quaternionic Kähler manifolds \mathcal{QM}_{4n+4} lying in the image of the c -map, irrespectively whether the pre-image Special Kähler manifold \mathcal{SK}_n is a symmetric space or not. The tri-holomorphic moment map of these isometries was presented in eq.s(4.27) and (4.31). The second Heisenberg algebra exists only in the case when the Quaternionic Kähler manifold \mathcal{QM}_{4n+4} is a symmetric space.

From this discussion we also realize that the central charge \mathbb{Z}_1 is just the L_+ generator of a universal $\mathfrak{sl}(2, R)_E$ Lie algebra that exists only in the symmetric space case and which was named the Ehlers algebra in the context of dimensional reduction analysis from $D = 4$ to $D = 3$ [71]. When $\mathfrak{sl}(2, R)_E$ does exist we can introduce the universal compact generator:

$$\mathfrak{S} \equiv L_+ - L_- = 2\lambda^2 \quad (4.56)$$

\mathcal{SK}_n Special Kähler manifold	\mathcal{QM}_{4n+4} Quaternionic Kähler manifold	$\dim \mathcal{SK}_n =$ n
$\frac{\mathrm{SU}(1,1)}{\mathrm{U}(1)}$	$\frac{\mathrm{G}_{2(2)}}{\mathrm{SU}(2) \times \mathrm{SU}(2)}$	$n = 1$
$\frac{\mathrm{Sp}(6, \mathbb{R})}{\mathrm{SU}(3) \times \mathrm{U}(1)}$	$\frac{\mathrm{F}_{4(4)}}{\mathrm{USp}(6) \times \mathrm{SU}(2)}$	$n = 6$
$\frac{\mathrm{SU}(3,3)}{\mathrm{SU}(3) \times \mathrm{SU}(3) \times \mathrm{U}(1)}$	$\frac{\mathrm{E}_{6(2)}}{\mathrm{SU}(6) \times \mathrm{SU}(2)}$	$n = 9$
$\frac{\mathrm{SO}^*(12)}{\mathrm{SU}(6) \times \mathrm{U}(1)}$	$\frac{\mathrm{E}_{7(-5)}}{\mathrm{SO}(12) \times \mathrm{SU}(2)}$	$n = 15$
$\frac{\mathrm{E}_{7(-25)}}{\mathrm{E}_{6(-78)} \times \mathrm{U}(1)}$	$\frac{\mathrm{E}_{8(-24)}}{\mathrm{E}_{7(-133)} \times \mathrm{SU}(2)}$	$n = 27$
$\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(2, 2+p)}{\mathrm{SO}(2) \times \mathrm{SO}(2+p)}$	$\frac{\mathrm{SO}(4, 4+p)}{\mathrm{SO}(4) \times \mathrm{SO}(4+p)}$	$n = 3 + p$
$\frac{\mathrm{SU}(p+1, 1)}{\mathrm{SU}(p+1) \times \mathrm{U}(1)}$	$\frac{\mathrm{SU}(p+2, 2)}{\mathrm{SU}(p+2) \times \mathrm{SU}(2)}$	$n = p + 1$

Table 1: List of special Kähler symmetric spaces with their Quaternionic Kähler c-map images. The number n denotes the complex dimension of the Special Kähler preimage. On the other hand $4n + 4$ is the real dimension of the Quaternionic Kähler c-map image.

which rotates the two sets of Heisenberg translations one into the other:

$$[\mathfrak{S}, \mathbf{W}^{i\alpha}] = \epsilon^{ij} \mathbf{W}^{j\alpha} \quad (4.57)$$

As we shall see, the gauging of this generator is a rather essential ingredient in the inclusion of one-field cosmological models into gauged $\mathcal{N} = 2$ supergravity.

The embedding tensor formulation of the gauging. It is useful to encode the choice of the gauge algebra in an *embedding tensor* $\theta_\Lambda{}^A$ [72, 73, 74, 75] through which the gauge generators are expressed in terms

of the global symmetry ones. If we denote by $\{t_A\} \equiv \{T^a, L^x, \mathbf{W}^{i\alpha}\}$ the generators of \mathbb{U}_Q and by X_Λ the gauge generators, since we are gauging only (abelian) isometries of the quaternionic manifold, we can write:

$$X_\Lambda = \theta_\Lambda^A t_A, \quad (4.58)$$

where the index Λ runs over all the vector fields. Of these only a subset will actually gauge the chosen isometries. This subset will be labelled by boldface latin indices $\mathbf{I}, \mathbf{J}, \dots$. The only condition on this tensor originates from the structure of the algebra:

$$[X_\Lambda, X_\Sigma] = 0 \Rightarrow \theta_\Lambda^A \theta_\Sigma^B f_{AB}^C = 0, \quad (4.59)$$

where f_{AB}^C are the structure constants of \mathbb{U}_Q . The triholomorphic moment maps \mathcal{P}_Λ^x and the Killing vectors k_Λ^u can then be expressed in the following way:

$$\mathcal{P}_\Lambda^x = \theta_\Lambda^A \mathcal{P}_A^x; \quad k_\Lambda^u = \theta_\Lambda^A k_A^u, \quad (4.60)$$

where \mathcal{P}_A^x, k_A^u are the *intrinsic* moment maps and Killing vectors associated with the quaternionic isometries. In order to make the analysis independent of the initial symplectic frame of the vector multiplet sector, it is useful to describe the gauge algebra generators by the (redundant) symplectic notation $X_M = (X_\Lambda, X^\Lambda) = \theta_M^A t_A$, see [75]. The tensor θ_M^A should then satisfy the locality constraint:

$$\theta_\Lambda^A \theta^\Lambda{}^B - \theta_\Lambda^B \theta^\Lambda{}^A = 0, \quad (4.61)$$

which guarantees that the tensor can be rotated, by means of a symplectic transformation, to an *electric frame* in which $\theta^\Lambda{}^A = 0$. For the restricted kind of gauging that we shall be dealing with, by extending the arguments given in [1], we can work in the electric frame to start with, with no loss of generality.

4.3.1 THE TRI-HOLOMORPHIC MOMENT MAP IN HOMOGENEOUS SYMMETRIC QUATERNIONIC KÄHLER MANIFOLDS

In the case the Quaternionic Kähler manifold \mathcal{QM}_{4n+4} is a homogeneous symmetric space $\frac{\mathbb{U}_Q}{\mathbb{H}_Q}$, the triholomorphic moment map associated with any generator of $\mathfrak{t} \in \mathbb{U}_Q$ of the isometry Lie algebra can be easily constructed by means of the formula:

$$\mathcal{P}_\mathfrak{t}^x = \text{Tr}_{[\mathbf{fun}]} (J^x \mathbb{L}_{Solv}^{-1} \mathfrak{t} \mathbb{L}_{Solv}) \quad (4.62)$$

where:

- a) J^x are the three generators of the $\mathfrak{su}(2)$ factor in the isotropy subalgebra $\mathbb{H} = \mathfrak{su}(2) \oplus \mathbb{H}'$, satisfying the quaternionic algebra (4.21). They should be normalized in such a way as to realize the following condition. Naming:

$$\Xi = \mathbb{L}_{Solv}^{-1}(q) d\mathbb{L}_{Solv}(q) \quad (4.63)$$

the Maurer Cartan differential one-form its projection on J^x should precisely yield the $\mathfrak{su}(2)$ one-form defined in eq. (4.13):

$$\omega = -\frac{i}{\sqrt{2}N_f} \sum_{x=1}^3 \text{Tr}_{[\mathbf{fun}]} (J^x \Xi) \gamma_x = \begin{pmatrix} -\frac{i}{2} \mathcal{Q} - \frac{i}{4} e^{-U} \Phi & e^{-\frac{U}{2}} V^T \mathbb{C} d\mathbf{Z} \\ -e^{-\frac{U}{2}} \bar{V}^T \mathbb{C} d\mathbf{Z} & \frac{i}{2} \mathcal{Q} + \frac{i}{4} e^{-U} \Phi \end{pmatrix} \quad (4.64)$$

In the above equation, which provides the precise link between the c -map description and the coset manifold description of the same geometry, $N_f = \dim \mathbf{fun}$ denotes the dimension of the fundamental representation of \mathbb{U}_Q .

b) The solvable coset representative $\mathbb{L}_{Solv}(q)$ is obtained by exponentiation of the Solvable Lie algebra:

$$\mathbb{L}_{Solv}(q) \simeq \exp \left[q \cdot Solv \left(\frac{U_Q}{\mathcal{H}_Q} \right) \right] \quad (4.65)$$

but the detailed exponentiation rule has to be determined in such a way that projecting the same Maurer Cartan form (4.63) along an appropriate basis of generators $T_{I|Solv}$ of the solvable Lie algebra $Solv \left(\frac{U_Q}{\mathcal{H}_Q} \right)$ we precisely obtain the vielbein E_{QM}^I defined in eq.(4.40). This summarized in the following general equations:

$$\begin{aligned} E_{QM}^I &= \text{Tr}_{[\text{fun}]} (T_{Solv}^I \Xi) \\ \delta_J^I &= \text{Tr}_{[\text{fun}]} (T_{Solv}^I T_{I|Solv}) \\ \Xi &= E_{QM}^I T_{I|Solv} \end{aligned} \quad (4.66)$$

In eq.(4.66) by T_{Solv}^I we have denoted the conjugate (with respect to the trace) of the solvable Lie algebra generators.

A general comment is in order. The precise calibration of the basis of the solvable generators T_{Solv}^I and of their exponentiation outlined in eq.(4.65) which allows the identification (4.66) is a necessary and quite laborious task in order to establish the bridge between the general c -map description of the quaternionic geometry and its actual realization in each symmetric coset model. This is also an unavoidable step in order to give a precise meaning to the very handy formula (4.62) for the tri-holomorphic map. It should also be noted that although (4.62) covers all the cases, the result of such a purely algebraic calculation is difficult to be guessed a priori. Hence educated guesses on the choice of generators whose gauging produces a priori determined features are difficult to be inferred from (4.62). The analytic structure of the tri-holomorphic moment map instead is much clearer in the c -map framework of formulae (4.27,4.31,4.36). The use of both languages and the construction of the precise bridge between them in each model is therefore an essential ingredient to understand the nature and the properties of candidate gaugings in whatever physical application.

Part III

ABELIAN GAUGINGS AND GENERAL PROPERTIES OF THEIR POTENTIALS IN THE c -MAP FRAMEWORK

As we stressed in the introduction the inclusion into $\mathcal{N} = 2$ supergravity obtained in [1] of inflaton potentials such as the Starobinsky potential⁶

$$V_{Starobinsky}(\phi) \equiv (1 - \exp[-\phi])^2 \quad (4.67)$$

is not occasional and limited to the case of hypermultiplets lying in $\frac{G(2,2)}{SU(2) \times SU(2)}$, rather it follows a general pattern that can be uncovered and relies on the properties of the c -map. In this way the mechanisms of the [1] can be generalized to larger Quaternionic Kähler manifolds opening a quite interesting new playground for the search of inflaton potentials that can be classified and understood in their geometrical origin.

Let us schematically summarize the main ingredients of the approach pioneered in [1] whose generalization we pursue in this paper:

- A) The inflaton field ϕ is assumed to belong to the hypermultiplet Quaternionic Kähler manifold \mathcal{QM} .
- B) In analogy with the construction in [1], we require the graviphoton not to be minimally coupled to any other field. This condition originally followed from the general argument that in the dual to the $R + R^2$ supergravity the central charge is gauged. This will amount to a constraint on the form of the embedding tensor θ defining the gauge algebra.
- C) The inflaton potential is generated by the gauging of an abelian subalgebra $\mathcal{A} \subset \text{iso}[\mathcal{QM}]$ of the isometry algebra of the hypermultiplet manifold.
- D) Since \mathcal{A} is abelian it is not required to have any action on the vector multiplet scalars ω^i which are inert. Actually it is quite desirable that the potential $V_{gauging}$ generated by the gauging allows to fix all the ω^i to their values at some reference point, say $\omega^i = 0$:

$$\left. \frac{\partial}{\partial \omega^i} V_{gauging} \right|_{\omega^i=0} = 0 \quad (4.68)$$

As shown in [1], one can generically guarantee the fixing conditions (4.68) if the Special Kähler Geometry of the vector multiplets is chosen to be that of the so named Minimal Coupling, defined below in eq.s (5.1-5.3).

- E) With the above choice of the vector multiplet geometry, after fixing the scalars ω^i the effective potential reduces to a sum of squares of the tri-holomorphic moment maps $P_{\mathcal{A}}^x$ which still depend on the variables $\{Z, U, a, z^i, \bar{z}^{i*}\}$. In order to approach effective potentials recognizable also as $\mathcal{N} = 1$ supergravity potentials one would like to be able to fix all the Heisenberg fields \mathbf{Z} (and possibly also the other fields U and a) to zero, remaining only with the complex fields (z^i, \bar{z}^{i*}) of the inner Special Kähler manifold. Looking at the general form (4.27) of the tri-holomorphic moment map for the Heisenberg algebra

⁶Just as in [1] we mention scalar fields that typically have non canonical kinetic terms

generators and (4.36) for the tri-holomorphic moment map of the inner Special Kähler isometries we immediately realize that, gauging these isometries *separately*, the condition:

$$\left. \frac{\partial}{\partial \mathbf{Z}^\alpha} \sum_{t \in \mathcal{A}} (\mathcal{P}_t)^2 \right|_{\mathbf{Z}=0} = 0 \quad (4.69)$$

is always satisfied. A gauge generator which is a combination of a translation in the Heisenberg algebra and a Special Kähler isometry, yields in general a scalar potential exhibiting linear terms in \mathbf{Z} , so that (4.69) provides a non-trivial constraint.

The definition of the locus \mathcal{L} involves setting to zero a certain number of fields ϕ^r belonging to \mathcal{SK}_n so that we should also realize the consistency condition:

$$\left. \frac{\partial}{\partial \phi^r} \sum_{t \in \mathcal{A}} (\mathcal{P}_t)^2 \right|_{\substack{\mathbf{Z} = 0 \\ \phi^r = 0}} = 0. \quad (4.70)$$

As mentioned earlier, the gauging yielding Starobinsky-like potentials need also involve the compact generator \mathfrak{S} . As we shall show in the following, if the gauged isometry is a combination of \mathfrak{S} and an \mathcal{SK}_n isometry, (4.69) poses no constraint on the gauging.

- F)** A favorite, though not mandatory, choice corresponds to looking for abelian generators of $\text{iso}[\mathcal{SK}_n]$ such that the locus which satisfies conditions (4.70) is defined by setting to zero all the axions p_r , namely all the fields associated with nilpotent generators of the solvable Lie algebra of \mathcal{SK}_n . The inclusion of the Starobinsky potential in supergravity was obtained in [1] precisely in this way. In section 8.4 we show a generalization of the same mechanism in the case of a bigger manifold \mathcal{QM}_{4n+4} , obtaining what can be denominated a multi Starobinsky model.
- G) The U -problem.** If we use only the type of isometries yielding the tri-holomorphic moment maps (4.27), (4.31) and (4.36) we face a serious problem with the fields U . It appears only through exponentials all of the same sign ($\exp[-2U]$ or $\exp[-U]$ in front of perfect squares. Hence the field U cannot be stabilized unless all such squares are zero which means no residual potential. To overcome such a problem one should have moment maps with the opposite sign of U in the exponential and this can happen only by introducing in the gauging either L_-^E or generators $\mathbf{W}^{2,\alpha}$ this means that such generators should exist, namely the manifold \mathcal{QM}_{4n+4} should be a symmetric space. In [1] the U -problem was solved by adding to a parabolic generator of a \mathcal{SK}_n -isometry the universal compact generator (4.56). As we have emphasized the Ehlers subalgebra exists in all symmetric spaces and so does the compact generator (4.56). This implies that the mechanism leading to the inclusion of the Starobinsky model found in [1] is actually rather universal and can be generalized in several ways.

The above discussion provides a framework for the search of other inflaton potentials.

5 MINIMAL COUPLING SPECIAL GEOMETRY

In this section we shortly describe the structure of the Minimal Coupling Special Kähler manifold \mathcal{MSK}_{p+1} , mostly in order to fix our conventions and to establish our notations. As announced in the introduction, this

kind of Special Geometry is our favorite choice for the vector multiplet sector of the $\mathcal{N} = 2$ lagrangian which allows us to construct an entire class of theories where the vector multiplet scalars can be stabilized and the effective potential of an abelian gauging is reduced only to the hypermultiplet sector. In view of such a use of \mathcal{MSK}_{p+1} , all items of its Special Geometry will be denoted with a hat, and its complex coordinates will be named ω_i rather than z^i . However it is clear that \mathcal{MSK}_{p+1} might also be used as c -map preimage of a Quaternionic Kähler manifold describing hypermultiplets.

As a manifold \mathcal{MSK}_{p+1} is the following coset:

$$\mathcal{MSK}_{p+1} = \frac{\text{SU}(1, p+1)}{\text{U}(1) \times \text{SU}(p+1)} \quad (5.1)$$

In terms of the complex coordinates ω^i a convenient choice of the $(2p+4)$ -dimensional holomorphic symplectic section is the following one:

$$\widehat{\Omega} = \left(\frac{\widehat{X}^\Lambda}{\widehat{F}_\Sigma} \right) = \begin{pmatrix} 1 \\ \omega^i \\ -i \\ i\omega^i \end{pmatrix} ; \quad (i = 1, \dots, p+1) \quad (5.2)$$

which leads to the following Kähler potential:

$$\widehat{\mathcal{K}} = -\log \left[-i\widehat{\Omega} \widehat{\mathbb{C}} \widehat{\Omega} \right] = -\log [2 (1 - \omega \cdot \bar{\omega})] \quad (5.3)$$

and to the following Kähler metric:

$$\widehat{g}_{ij^*} = \partial_i \partial_{j^*} \widehat{\mathcal{K}} = \frac{1}{(1 - \omega \cdot \bar{\omega})^2} (\delta^{ij} (1 - \omega \cdot \bar{\omega}) + \bar{\omega}^i \omega^j) \quad (5.4)$$

Defining the Kähler covariant derivatives of the covariantly holomorphic sections as in eq.s (2.22) we obtain three results that are very important for the discussion of reduced scalar potentials in the present paper. Firstly we get:

$$\nabla_i \widehat{U}_j \equiv \nabla_i \nabla_j \widehat{V} = 0 \quad (5.5)$$

which compared with eq.(2.26) implies the vanishing of the three-index symmetric tensor \widehat{C}_{ijk} . This unique property of the special Kähler manifold \mathcal{MSK}_{p+1} defined by eq.(5.1) is the reason why it has been named the Minimal Coupling Special Geometry, the interpretation of the tensor C_{ijk} in phenomenological applications being that of Yukawa couplings of the gauginos. In ref. [1] it was shown that the vanishing of \widehat{C}_{ijk} guarantees the consistency (see eq. (3.10) of the quoted reference) of the truncation of the classical supergravity theory to the hypermultiplet quaternionic scalars by fixing the vector multiplet scalars to the origin of their manifold:

$$\omega^i = 0 \quad (5.6)$$

Secondly we evaluate the the covariantly symplectic holomorphic section in the origin of the manifold and we obtain:

$$\widehat{V}|_{\omega=0} = \frac{1}{\sqrt{2}} \left\{ \begin{array}{c|c} 1 & 0 \\ \hline -i & 0 \end{array} \right\} \quad (5.7)$$

In the same point we have:

$$\left(\widehat{g}^{ij*} \nabla_i \widehat{V}^\alpha \nabla_{j*} \widehat{V} \right) \Big|_{\omega=0} = \frac{1}{2} \left(\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & \mathbf{1}_{(p+1) \times (p+1)} & 0 & i \mathbf{1}_{(p+1) \times (p+1)} \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & -i \mathbf{1}_{(p+1) \times (p+1)} & 0 & \mathbf{1}_{(p+1) \times (p+1)} \end{array} \right) \quad (5.8)$$

5.1 GAUGING ABELIAN ISOMETRIES OF THE HYPERMULTIPLETS

Relying on these results we see that if the hypermultiplet Quaternionic manifold \mathcal{QM}_{4m} possesses a $p+1$ -dimensional abelian Lie algebra of isometries, we can always gauge them by using, for the vector multiplets, the Special Kähler manifold \mathcal{MSK}_{p+1} introducing also the following embedding tensor:

$$\theta_M^I \equiv \left\{ \theta_\Lambda^I, \theta^{\Sigma|I} \right\} = \left\{ \theta_0^I = 0, \theta_J^I = \delta_J^I, \theta^{\Sigma|I} = 0 \right\}. \quad (5.9)$$

Notice that the choice of setting $\theta_0^I = 0$ follows from the requirement B) that the graviphoton should not be gauged. This indeed amounts to requiring:

$$\widehat{V} \Big|_{\omega=0}^M \theta_M^{\mathcal{A}} = 0 \Rightarrow \theta_0^I = 0. \quad (5.10)$$

In such a theory the scalar potential has the following general form:

$$\mathcal{V}_{scalar}(\omega, \bar{\omega}, q) = 4 k_I^u k_J^v h_{uv} \widehat{V}^I \widehat{V}^J + \left(\widehat{g}^{ij*} \nabla_i \widehat{V}^I \nabla_{j*} \widehat{V}^J - 3 \widehat{V}^I \widehat{V}^J \right) \mathcal{P}_I^x \mathcal{P}_J^x \quad (5.11)$$

setting $\omega^i = 0$ is a consistent truncation and the reduced potential takes the following universal general form which is positive definite by construction:

$$\mathcal{V}_{scalar}(0, 0, q) = \sum_{I=1}^{p+1} \mathcal{P}_I^x(q) \mathcal{P}_I^x(q) \quad (5.12)$$

In the next section 5.2 we reconsider the derivation of the Starobinsky potential obtained in [1] from a parabolic gauging as a master example that can be generalized to bigger manifolds.

5.2 THE STAROBINSKY POTENTIAL

Last year a great deal of activity was devoted to the inclusion of phenomenologically interesting inflaton potentials into $\mathcal{N} = 1$ supergravity as we recalled in the introduction. A first wave of investigations considered the possible generation of potentials by means of suitably chosen superpotentials, subsequently, after an important new viewpoint was introduced in [30] and was subsequently developed in [33],[34],[39],[40], [41],[42], it became clear that positive definite inflaton potentials can be generated by the gauging of some isometry of the Kähler manifold of scalar multiplets. Such potentials have the form of squares of Kähler moment maps. In [41] this mechanism was applied to the case of constant curvature one-dimensional Kähler manifolds and it was shown that Starobinsky-like potentials [10] emerge from the moment map of a parabolic isometry in $\text{SL}(2, \mathbb{R}) \simeq \text{SU}(1, 1)$ with the addition of a Fayet Iliopoulos term. In particular the standard Starobinsky

model that is dual to an $R + R^2$ supergravity emerges from gauging the parabolic shift isometry of an $\frac{SU(1,1)}{U(1)}$ manifold with Kähler potential $\mathcal{K} = -3 \log(z - \bar{z})$ which is precisely the Special Kähler manifold S^3 . Let us now consider eq.(4.36) and we can learn an important lesson. If in the c -map image of some \mathcal{SK} Special Kähler manifold, for instance the S^3 model, we gauge, according to the scheme discussed in section III, some nilpotent Lie algebra element $\mathfrak{N}_+ \in \mathbb{U}_{\mathcal{SK}} \subset \mathbb{U}_{\mathcal{Q}}$ identical with the parabolic shift generator that we would have gauged in $\mathcal{N} = 1$ supergravity, (for instance the generator $L_+ \in \mathfrak{sl}(2, \mathbb{R})$ in the case of the S^3 model), we obtain a moment map that contains precisely the $\mathcal{P}_{\mathbf{I}}$ of the $\mathcal{N} = 1$ case, modified by \mathbf{Z} dependent terms. In case the \mathbf{Z} can be stabilized to zero the remaining effective potential is that of the corresponding $\mathcal{N} = 1$ theory, apart from the Fayet Iliopoulos term. There are two remaining problems. The generation of a Fayet Iliopoulos term and the stabilization of the U field. They are solved in one stroke by modifying the parabolic generator of the inner Special Kähler isometry with the addition of the universal Ehlers rotation (4.56).

Let us see how this works.

With reference to eq.s (6.35) let us consider the following generator:

$$\mathfrak{p} = \mathfrak{N}_+ + \kappa \mathfrak{S} \quad (5.13)$$

where \mathfrak{N}_+ is the previously mentioned nilpotent element of the Special Kähler subalgebra ($\mathfrak{N}_+^r = 0$, for some positive integer r) and κ is a parameter. Let us then calculate the tri-holomorphic moment map $\mathcal{P}_{\mathfrak{p}}^x$ according to formula (6.41).

Because of the linearity of the momentum map in Lie algebra elements we have:

$$\begin{aligned} \mathfrak{P}_{\mathfrak{p}} &= \mathfrak{P}_{\mathfrak{N}_+} + \mathfrak{P}_{\mathfrak{S}} \\ \mathfrak{P}_{\mathfrak{N}_+} &= \left(\begin{array}{c|c} \frac{i}{4} \mathcal{P}_{\mathfrak{N}_+} + \mathcal{O}(\mathbf{Z}^2) & \mathcal{O}(\mathbf{Z}) \\ \hline \mathcal{O}(\mathbf{Z}) & -\frac{i}{4} \mathcal{P}_{\mathfrak{N}_+} - \mathcal{O}(\mathbf{Z}^2) \end{array} \right) \\ \mathfrak{P}_{\mathfrak{S}} &= \left(\begin{array}{c|c} \frac{i}{8} e^{-U} (1 + a^2 + e^{2U}) + \mathcal{O}(\mathbf{Z}^2) & \mathcal{O}(\mathbf{Z}) \\ \hline \mathcal{O}(\mathbf{Z}) & -\frac{i}{8} e^{-U} (1 + a^2 + e^{2U}) - \mathcal{O}(\mathbf{Z}^2) \end{array} \right) \end{aligned} \quad (5.14)$$

where $\mathcal{P}_{\mathfrak{N}_+}$ is the Kählerian moment map of the Killing vector associated with the generator \mathfrak{N}_+ as defined in eq.(2.55). It is evident by the above completely universal formulae that the potential:

$$V_{gauging} = \text{const Tr} [\mathfrak{P}_{\mathfrak{p}} \cdot \mathfrak{P}_{\mathfrak{p}}] \quad (5.15)$$

possesses the following universal property:

$$\left. \frac{\partial}{\partial \mathbf{Z}^\alpha} V_{gauging} \right|_{\mathbf{Z}=0} = 0 \quad (5.16)$$

allowing for a consistent truncation of the Heisenberg fields. After such truncation we find:

$$V_{eff}(U, a, z, \bar{z}) = V_{gauging}|_{\mathbf{Z}=0} = \text{const} \times \left[\mathcal{P}_{\mathfrak{N}_+} + \frac{\kappa}{2} e^{-U} (1 + a^2 + e^{2U}) \right]^2 \quad (5.17)$$

From equation (5.17) we further learn that we can consistently truncate the fields a and U setting them to zero since

$$\left. \frac{\partial}{\partial U} V_{eff} \right|_{U=a=0} = 0 \quad ; \quad \left. \frac{\partial}{\partial a} V_{eff} \right|_{U=a=0} = 0 \quad (5.18)$$

We find:

$$V_{infl}(z, \bar{z}) \equiv V_{eff}(0, 0, z, \bar{z}) = (\mathcal{P}_{\mathfrak{N}_+} + \kappa)^2 \quad (5.19)$$

which clearly shows how the universal generator \mathfrak{S} provides, after stabilization of the U field, the mechanism that generates the Fayet Iliopoulos term [45] essential for inflation.

Part IV

EXAMPLES

As an illustration of the general patterns and mechanisms described in the previous pages we consider two examples of Quaternionic Kähler manifolds \mathcal{QM}_{4n+4} obtained from the c -map of two homogeneous symmetric Special Kähler manifolds \mathcal{SK}_n .

1. The manifold $\frac{G_{(2,2)}}{SU(2) \times SU(2)}$ which is the c -map image of the Special Kähler manifold $\frac{SU(1,1)}{U(1)}$ with cubic embedding of $SU(1,1)$ in $Sp(4, \mathbb{R})$. In this case $n = 1$ and the corresponding coupling of one vector multiplet to supergravity is usually named the S^3 model in the literature.
2. The manifold $\frac{F_{(4,4)}}{SU(2) \times USp(6)}$ which is the c -map image of the Special Kähler manifold $\frac{Sp(6, \mathbb{R})}{SU(3) \times U(1)}$. In this case $n = 6$.

For these two models we provide a full fledged construction of all the geometrical items and in particular we realize the bridge between the algebraic description and the analytic one advocated at the end of section 4.3.1. This allows us to discuss a couple of examples of gaugings. In particular in the case of the of the first model which we utilized as a calibration device for our general formulae we retrieve the inclusion of the Starobinsky model first demonstrated in [1].

The detailed construction of the second model, which we plan to utilize in future publications for an extensive and possibly exhaustive analysis of gaugings in larger hypermultiplet spaces, is utilized in the present paper to provide an example of generalization of the results of [1] by means of the inclusion of a multi Starobinsky model.

6 THE S^3 MODEL AND ITS QUATERNIONIC IMAGE $\frac{G_{(2,2)}}{SU(2) \times SU(2)}$

In this which is the simplest example $n = 1$, namely the Special Kähler manifold has complex dimension 1 and it can be identified with the time honored Poincaré Lobachevsky plane:

$$\mathcal{SK}_1 = \frac{SU(1,1)}{U(1)} \quad (6.1)$$

6.1 THE SPECIAL KÄHLER STRUCTURE OF S^3

The corresponding Kähler potential is:

$$\mathcal{K} = -\log [(z - \bar{z})^3] \quad (6.2)$$

which leads to the Kähler metric:

$$g_{z\bar{z}} = \frac{3}{4} \frac{1}{(z - \bar{z})^2} \quad (6.3)$$

Setting:

$$z = i \exp[h] + y \quad (6.4)$$

we get:

$$g_{z\bar{z}} = \frac{3}{2} (dh^2 + \exp[-2h] dy^2) \quad (6.5)$$

In the notations of [71] the holomorphic symplectic section governing this special geometry is given by the following four component vector:

$$\Omega = \left\{ -\sqrt{3}z^2, z^3, \sqrt{3}z, 1 \right\} \quad (6.6)$$

In this case the \mathbf{W} -representation is the spin $j = \frac{3}{2}$ of the $\text{SL}(2, \mathbb{R}) \sim \text{SU}(1, 1)$ group that happens to be four dimensional symplectic:

$$\text{SL}(2, \mathbb{R}) \ni \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \begin{pmatrix} da^2 + 2bca & -\sqrt{3}a^2c & -cb^2 - 2adb & -\sqrt{3}b^2d \\ -\sqrt{3}a^2b & a^3 & \sqrt{3}ab^2 & b^3 \\ -bc^2 - 2adc & \sqrt{3}ac^2 & ad^2 + 2bcd & \sqrt{3}bd^2 \\ -\sqrt{3}c^2d & c^3 & \sqrt{3}cd^2 & d^3 \end{pmatrix} \in \text{Sp}(4, \mathbb{R}) \quad (6.7)$$

the preserved symplectic metric being the following one:

$$\mathbb{C} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad (6.8)$$

According to the general rule the Kähler potential (6.2) is retrieved by setting:

$$\mathcal{K}(z, \bar{z}) = -\log \left[-i\Omega \mathbb{C} \bar{\Omega} \right] \quad (6.9)$$

6.2 THE MATRIX \mathcal{M}_4^{-1} AND THE c -MAP

For the S^3 model the matrix \mathcal{M}_4 and its inverse have the following explicit appearance:

$$\mathcal{M}_4 = \begin{pmatrix} \frac{4iz\bar{z}(z^2+4\bar{z}z+\bar{z}^2)}{(z-\bar{z})^3} & -\frac{4i\sqrt{3}z^2\bar{z}^2(z+\bar{z})}{(z-\bar{z})^3} & -\frac{i(z+\bar{z})(z^2+10\bar{z}z+\bar{z}^2)}{(z-\bar{z})^3} & -\frac{2i\sqrt{3}(z+\bar{z})^2}{(z-\bar{z})^3} \\ -\frac{4i\sqrt{3}z^2\bar{z}^2(z+\bar{z})}{(z-\bar{z})^3} & \frac{8iz^3\bar{z}^3}{(z-\bar{z})^3} & \frac{2i\sqrt{3}z\bar{z}(z+\bar{z})^2}{(z-\bar{z})^3} & \frac{i(z+\bar{z})^3}{(z-\bar{z})^3} \\ -\frac{i(z+\bar{z})(z^2+10\bar{z}z+\bar{z}^2)}{(z-\bar{z})^3} & \frac{2i\sqrt{3}z\bar{z}(z+\bar{z})^2}{(z-\bar{z})^3} & \frac{4i(z^2+4\bar{z}z+\bar{z}^2)}{(z-\bar{z})^3} & \frac{4i\sqrt{3}(z+\bar{z})}{(z-\bar{z})^3} \\ -\frac{2i\sqrt{3}(z+\bar{z})^2}{(z-\bar{z})^3} & \frac{i(z+\bar{z})^3}{(z-\bar{z})^3} & \frac{4i\sqrt{3}(z+\bar{z})}{(z-\bar{z})^3} & \frac{8i}{(z-\bar{z})^3} \end{pmatrix} \quad (6.10)$$

its inverse being:

$$\mathcal{M}_4^{-1} = \begin{pmatrix} \frac{4i(z^2+4\bar{z}z+\bar{z}^2)}{(z-\bar{z})^3} & \frac{4i\sqrt{3}(z+\bar{z})}{(z-\bar{z})^3} & \frac{i(z^3+11\bar{z}z^2+11\bar{z}^2z+\bar{z}^3)}{(z-\bar{z})^3} & -\frac{2i\sqrt{3}z\bar{z}(z+\bar{z})^2}{(z-\bar{z})^3} \\ \frac{4i\sqrt{3}(z+\bar{z})}{(z-\bar{z})^3} & \frac{8i}{(z-\bar{z})^3} & \frac{2i\sqrt{3}(z+\bar{z})^2}{(z-\bar{z})^3} & -\frac{i(z+\bar{z})^3}{(z-\bar{z})^3} \\ \frac{i(z^3+11\bar{z}z^2+11\bar{z}^2z+\bar{z}^3)}{(z-\bar{z})^3} & \frac{2i\sqrt{3}(z+\bar{z})^2}{(z-\bar{z})^3} & \frac{4iz\bar{z}(z^2+4\bar{z}z+\bar{z}^2)}{(z-\bar{z})^3} & -\frac{4i\sqrt{3}z^2\bar{z}^2(z+\bar{z})}{(z-\bar{z})^3} \\ -\frac{2i\sqrt{3}z\bar{z}(z+\bar{z})^2}{(z-\bar{z})^3} & -\frac{i(z+\bar{z})^3}{(z-\bar{z})^3} & -\frac{4i\sqrt{3}z^2\bar{z}^2(z+\bar{z})}{(z-\bar{z})^3} & \frac{8iz^3\bar{z}^3}{(z-\bar{z})^3} \end{pmatrix} \quad (6.11)$$

Furthermore, in this case a convenient reference point is given by $z_0 = i$ that can be mapped into any point of the upper complex plane by means of the element:

$$\mathfrak{g}_z = \begin{pmatrix} e^{h/2} & e^{-h/2}y \\ 0 & e^{-h/2} \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \quad (6.12)$$

acting by means of fractional linear transformations. The explicit form of the $\Lambda(\mathbf{g})$ matrix in the \mathbf{W} -representation was given in eq.(6.7). This provides us with all the necessary information in order to write down the explicit form of the $E_{\mathcal{QM}}^I$ vielbein for the S^3 case.

6.3 THE VIELBEIN AND THE BORELLIAN MAURER CARTAN EQUATIONS

They are the following ones:

$$E_{\mathcal{QM}}^I = \frac{1}{2} \begin{pmatrix} dU \\ \sqrt{3}dh \\ \sqrt{3}dye^{-h} \\ e^{-U} (da + dZ_3 Z_1 + dZ_4 Z_2 - dZ_1 Z_3 - dZ_2 Z_4) \\ \sqrt{2}e^{-\frac{h}{2}-\frac{U}{2}} (dZ_1 + y (2dZ_3 - \sqrt{3}y dZ_4)) \\ \sqrt{2}e^{-\frac{3h}{2}-\frac{U}{2}} ((\sqrt{3}dZ_3 - y dZ_4) y^2 + \sqrt{3}dZ_1 y + dZ_2) \\ \sqrt{2}e^{\frac{h-U}{2}} (dZ_3 - \sqrt{3}y dZ_4) \\ \sqrt{2}e^{\frac{3h}{2}-\frac{U}{2}} dZ_4 \end{pmatrix} \quad (6.13)$$

Furthermore we find $\mathcal{M}_4^{-1}(\mathbf{i}, -\mathbf{i}) = -\mathbf{1}_{4 \times 4}$ so that the quadratic form (4.42) is just:

$$\mathbf{q}_{AB} = \text{diag} (1, 1, 1, 1, 1, 1, 1, 1) \quad (6.14)$$

The next step consists of calculating the geometry of the space described by the above vielbein and flat metric (6.14). To this effect we have first to calculate the torsion, namely the exterior derivatives of the vielbein and then using such a result the spin connection ω^{IJ} , finally the curvature two-form from which we extract the Riemann and the Ricci tensor.

Addressing the first step, namely the contorsion, we have the first important surprise. The exterior derivatives of the vielbein are expressed in terms of wedge-quadratic products of the same vielbein with constant numerical coefficients. This means that the above constructed vielbein satisfy a set of Maurer Cartan equations describing a Lie algebra, namely⁷:

$$dE^I - \frac{1}{2} f_{JK}{}^I E^J \wedge E^K = 0 \quad (6.15)$$

the tensor $f_{BC}{}^A$ being the structure constants of such a Lie algebra. Explicitly for the S^3 model we get:

$$\begin{aligned} 0 &= dE^1 \\ 0 &= dE^2 \\ 0 &= dE^3 + 2 \frac{E^2 \wedge E^3}{\sqrt{3}} \\ 0 &= dE^4 + 2 E^1 \wedge E^4 - 2 E^5 \wedge E^7 - 2 E^6 \wedge E^8 \\ 0 &= dE^5 + E^1 \wedge E^5 + \frac{E^2 \wedge E^5}{\sqrt{3}} - \frac{4 E^3 \wedge E^7}{\sqrt{3}} \\ 0 &= dE^6 + E^1 \wedge E^6 + \sqrt{3} E^2 \wedge E^6 - 2 E^3 \wedge E^5 \\ 0 &= dE^7 + E^1 \wedge E^7 - \frac{E^2 \wedge E^7}{\sqrt{3}} + 2 E^3 \wedge E^8 \\ 0 &= dE^8 + E^1 \wedge E^8 - \sqrt{3} E^2 \wedge E^8 \end{aligned} \quad (6.16)$$

⁷Note that here, for simplicity we have dropped the suffix \mathcal{SK} . This is done for simplicity since there is no risk of confusion.

Hence it arises the following question: which Lie algebra is described by such Maurer Cartan equations? Utilizing the standard method of diagonalizing the adjoint action of the two commuting generators $H_{1,2}$ dual to $E^{1,2}$ we find that the eigenvalues are just the positive roots of $\mathfrak{g}_{2,2}$:

$$\begin{aligned} \alpha_1 &= (1, 0) & ; & & \alpha_2 &= \frac{\sqrt{3}}{2}(-\sqrt{3}, 1) \\ \alpha_3 &= \alpha_1 + \alpha_2 &= \frac{1}{2}(-1, \sqrt{3}) & ; & \alpha_4 &= 2\alpha_1 + \alpha_2 &= \frac{1}{2}(1, \sqrt{3}) \\ \alpha_5 &= 3\alpha_1 + \alpha_2 &= \frac{\sqrt{3}}{2}(\sqrt{3}, 1) & ; & \alpha_6 &= 3\alpha_1 + 2\alpha_2 &= (0, \sqrt{3}) \end{aligned} \quad (6.17)$$

As it is well known the complex Lie algebra $\mathfrak{g}_2(\mathbb{C})$ has rank two and it is defined by the 2×2 Cartan matrix encoded in the following Dynkin diagram:

$$\mathfrak{g}_2 \quad \text{---} \circ \rightrightarrows \circ \quad = \quad \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

The real form $\mathfrak{g}_{2,2}$ is the maximally split form of the above complex Lie algebra. With a little bit of more work we can put eq.s(6.16) into the standard Cartan Weyl form for the Borel subalgebra of $\mathfrak{g}_{2,2}$, composed by the Cartan generators and by all the positive root step operators. Naming T_J the generators dual to the vielbein E^I such that $E^I(T_J) = \delta_J^I$, we find that the appropriate identifications are the following ones:

$$\begin{aligned} T_2 &= 2 \frac{\mathcal{H}_1}{\sqrt{3}} \\ T_1 &= 2 \frac{\mathcal{H}_2}{\sqrt{3}} \\ T_3 &= 2 E^{\alpha_1} \\ T_4 &= 2 E^{\alpha_6} \\ T_8 &= 2 E^{\alpha_2} \\ T_7 &= 2 E^{\alpha_3} \\ T_5 &= 2 E^{\alpha_4} \\ T_6 &= 2 E^{\alpha_5} \end{aligned} \quad (6.18)$$

We conclude that the manifold on which the metric (4.4) is constructed is homeomorphic to the solvable group-manifold $\text{Bor}(\mathfrak{g}_{2,2})$.

6.4 THE SPIN CONNECTION

Next, calculating the Levi-Civita spin connection from its definition, namely the vanishing torsion condition:

$$0 = dE^I + \omega^{IJ} \wedge E^J \quad (6.19)$$

we find the following result:

$$\omega^{IJ} = \begin{pmatrix} 0 & 0 & 0 & E^4 & \frac{E^5}{2} & \frac{E^6}{2} & \frac{E^7}{2} & \frac{E^8}{2} \\ 0 & 0 & \frac{E^3}{\sqrt{3}} & 0 & \frac{E^5}{2\sqrt{3}} & \frac{1}{2}\sqrt{3}E^6 & -\frac{E^7}{2\sqrt{3}} & -\frac{1}{2}\sqrt{3}E^8 \\ 0 & -\frac{E^3}{\sqrt{3}} & 0 & 0 & -\frac{E^6}{2} - \frac{E^7}{\sqrt{3}} & -\frac{E^5}{2} & \frac{E^8}{2} - \frac{E^5}{\sqrt{3}} & \frac{E^7}{2} \\ -E^4 & 0 & 0 & 0 & \frac{E^7}{2} & \frac{E^8}{2} & -\frac{E^5}{2} & -\frac{E^6}{2} \\ -\frac{E^5}{2} & -\frac{E^5}{2\sqrt{3}} & \frac{E^6}{2} + \frac{E^7}{\sqrt{3}} & -\frac{E^7}{2} & 0 & \frac{E^3}{2} & -\frac{E^3}{\sqrt{3}} - \frac{E^4}{2} & 0 \\ -\frac{E^6}{2} & -\frac{1}{2}\sqrt{3}E^6 & \frac{E^5}{2} & -\frac{E^8}{2} & -\frac{E^3}{2} & 0 & 0 & -\frac{E^4}{2} \\ -\frac{E^7}{2} & \frac{E^7}{2\sqrt{3}} & \frac{E^5}{\sqrt{3}} - \frac{E^8}{2} & \frac{E^5}{2} & \frac{E^3}{\sqrt{3}} + \frac{E^4}{2} & 0 & 0 & \frac{E^3}{2} \\ -\frac{E^8}{2} & \frac{1}{2}\sqrt{3}E^8 & -\frac{E^7}{2} & \frac{E^6}{2} & 0 & \frac{E^4}{2} & -\frac{E^3}{2} & 0 \end{pmatrix} \quad (6.20)$$

which can be decomposed in the way we now describe.

6.5 HOLONOMY ALGEBRA AND DECOMPOSITION OF THE SPIN CONNECTION

Let us introduce two triplets $J_{[I]}^x$ and $J_{[II]}^x$ of 8×8 matrices that can be read off explicitly as the coefficients of α_x and β_x in the following linear combinations:

$$\sum_{x=1}^3 \alpha_x J_{[I]}^x = \begin{pmatrix} 0 & 0 & 0 & -\frac{\alpha_1}{2} & -\frac{1}{4}\sqrt{3}\alpha_3 & -\frac{\alpha_2}{4} & \frac{\sqrt{3}\alpha_2}{4} & -\frac{\alpha_3}{4} \\ 0 & 0 & \frac{\alpha_1}{2} & 0 & -\frac{\alpha_3}{4} & -\frac{1}{4}\sqrt{3}\alpha_2 & -\frac{\alpha_2}{4} & \frac{\sqrt{3}\alpha_3}{4} \\ 0 & -\frac{\alpha_1}{2} & 0 & 0 & -\frac{\alpha_2}{4} & \frac{\sqrt{3}\alpha_3}{4} & \frac{\alpha_3}{4} & \frac{\sqrt{3}\alpha_2}{4} \\ \frac{\alpha_1}{2} & 0 & 0 & 0 & \frac{\sqrt{3}\alpha_2}{4} & -\frac{\alpha_3}{4} & \frac{\sqrt{3}\alpha_3}{4} & \frac{\alpha_2}{4} \\ \frac{\sqrt{3}\alpha_3}{4} & \frac{\alpha_3}{4} & \frac{\alpha_2}{4} & -\frac{1}{4}\sqrt{3}\alpha_2 & 0 & \frac{\sqrt{3}\alpha_1}{4} & -\frac{\alpha_1}{4} & 0 \\ \frac{\alpha_2}{4} & \frac{\sqrt{3}\alpha_2}{4} & -\frac{1}{4}\sqrt{3}\alpha_3 & \frac{\alpha_3}{4} & -\frac{1}{4}\sqrt{3}\alpha_1 & 0 & 0 & \frac{\alpha_1}{4} \\ -\frac{1}{4}\sqrt{3}\alpha_2 & \frac{\alpha_2}{4} & -\frac{\alpha_3}{4} & -\frac{1}{4}\sqrt{3}\alpha_3 & \frac{\alpha_1}{4} & 0 & 0 & \frac{\sqrt{3}\alpha_1}{4} \\ \frac{\alpha_3}{4} & -\frac{1}{4}\sqrt{3}\alpha_3 & -\frac{1}{4}\sqrt{3}\alpha_2 & -\frac{\alpha_2}{4} & 0 & -\frac{\alpha_1}{4} & -\frac{1}{4}\sqrt{3}\alpha_1 & 0 \end{pmatrix} \quad (6.21)$$

$$\sum_{x=1}^3 \beta_x J_{[II]}^x = \begin{pmatrix} 0 & 0 & 0 & \frac{3\beta_1}{2} & -\frac{1}{4}\sqrt{3}\beta_3 & -\frac{3\beta_2}{4} & -\frac{1}{4}\sqrt{3}\beta_2 & \frac{3\beta_3}{4} \\ 0 & 0 & \frac{\beta_1}{2} & 0 & -\frac{\beta_3}{4} & -\frac{3}{4}\sqrt{3}\beta_2 & \frac{\beta_2}{4} & -\frac{3}{4}\sqrt{3}\beta_3 \\ 0 & -\frac{\beta_1}{2} & 0 & 0 & \frac{5\beta_2}{4} & \frac{\sqrt{3}\beta_3}{4} & \frac{5\beta_3}{4} & -\frac{1}{4}\sqrt{3}\beta_2 \\ -\frac{3\beta_1}{2} & 0 & 0 & 0 & -\frac{1}{4}\sqrt{3}\beta_2 & \frac{3\beta_3}{4} & \frac{\sqrt{3}\beta_3}{4} & \frac{3\beta_2}{4} \\ \frac{\sqrt{3}\beta_3}{4} & \frac{\beta_3}{4} & -\frac{5\beta_2}{4} & \frac{\sqrt{3}\beta_2}{4} & 0 & \frac{\sqrt{3}\beta_1}{4} & -\frac{5\beta_1}{4} & 0 \\ \frac{3\beta_2}{4} & \frac{3\sqrt{3}\beta_2}{4} & -\frac{1}{4}\sqrt{3}\beta_3 & -\frac{3\beta_3}{4} & -\frac{1}{4}\sqrt{3}\beta_1 & 0 & 0 & -\frac{3\beta_1}{4} \\ \frac{\sqrt{3}\beta_2}{4} & -\frac{\beta_2}{4} & -\frac{5\beta_3}{4} & -\frac{1}{4}\sqrt{3}\beta_3 & \frac{5\beta_1}{4} & 0 & 0 & \frac{\sqrt{3}\beta_1}{4} \\ -\frac{3\beta_3}{4} & \frac{3\sqrt{3}\beta_3}{4} & \frac{\sqrt{3}\beta_2}{4} & -\frac{3\beta_2}{4} & 0 & \frac{3\beta_1}{4} & -\frac{1}{4}\sqrt{3}\beta_1 & 0 \end{pmatrix} \quad (6.22)$$

Both triplets form an 8-dimensional representation of the $\mathfrak{su}(2)$ Lie algebra and the two triplets commute with each other:

$$\begin{aligned} \left[J_{[I]}^x, J_{[I]}^y \right] &= \epsilon^{xyz} J_{[I]}^z \\ \left[J_{[II]}^x, J_{[II]}^y \right] &= \epsilon^{xyz} J_{[II]}^z \\ \left[J_{[I]}^x, J_{[II]}^y \right] &= 0 \end{aligned} \quad (6.23)$$

Furthermore all matrices are antisymmetric so that the two Lie algebras $\mathfrak{su}_I(2)$ and $\mathfrak{su}_{II}(2)$ are both subalgebras of $\mathfrak{so}(8)$. The distinction between these two representations becomes clear when we calculate the Casimir operator for both of them. We obtain:

$$\sum_{x=1}^3 J_{[I]}^x \cdot J_{[I]}^x = -\frac{3}{4} \mathbf{1} \quad ; \quad \sum_{x=1}^3 J_{[II]}^x \cdot J_{[II]}^x = -\frac{15}{4} \mathbf{1} \quad (6.24)$$

Hence the first $\mathfrak{su}_I(2)$ Lie algebra is realized on the considered eight-dimensional space in the $j = \frac{1}{2}$ representation, while the second $\mathfrak{su}_{II}(2)$ Lie algebra is realized on the same space in the $j = \frac{3}{2}$. In other words, with respect to both subalgebras of $\mathfrak{so}(8)$, the fundamental representation decomposes as follows:

$$\mathbf{8}^{\mathfrak{su}_I(2) \oplus \mathfrak{su}_{II}(2) \subset \mathfrak{so}(8)} \xrightarrow{\quad} (\mathbf{2}, \mathbf{4}) \quad (6.25)$$

By direct calculation we verify that the spin connection displayed in equation (6.20) has the following structure:

$$\omega = \omega_x^{[I]} J_{[I]}^x \oplus \omega_x^{[II]} J_{[II]}^x \quad (6.26)$$

where:

$$\omega_x^{[I]} = \begin{pmatrix} \sqrt{3}E^3 - E^4 \\ \sqrt{3}E^7 - E^6 \\ -E^8 - \sqrt{3}E^5 \end{pmatrix} \quad ; \quad \omega_x^{[II]} = \begin{pmatrix} \frac{E^4}{2} + \frac{E^3}{2\sqrt{3}} \\ -\frac{E^7}{2\sqrt{3}} - \frac{E^6}{2} \\ \frac{E^8}{2} - \frac{E^5}{2\sqrt{3}} \end{pmatrix} \quad (6.27)$$

This structure clearly demonstrates the reduced holonomy of the Quaternionic Kähler manifold. Indeed, according to eq.(6.25) the vielbein transforms in the doublet of $\mathfrak{su}_I(2)$ tensored with the fundamental representation of $\mathfrak{sp}(4, \mathbb{R})$. In the present case the symplectic 4×4 matrices are actually reduced to the subalgebra $\mathfrak{su}_{II}(2) \subset \mathfrak{sp}(4, \mathbb{R})$ with respect to which the fundamental of $\mathfrak{sp}(4, \mathbb{R})$ remains irreducible and coincides with the $j = \frac{3}{2}$ representation of $\mathfrak{su}_{II}(2)$. The above discussion can be summarized by the statement:

$$\mathfrak{so}(8) \subset \mathfrak{su}(2) \oplus \mathfrak{usp}(4) \subset \text{Hol} = \mathfrak{su}_I(2) \oplus \mathfrak{su}_{II}(2) \quad (6.28)$$

by definition the holonomy algebra being the Lie algebra in which the Levi-Civita spin connection takes values.

6.6 STRUCTURE OF THE ISOTROPY SUBALGEBRA \mathbb{H}

It remains to single out the structure of the denominator subalgebra $\mathbb{H} \subset \mathbb{U} \equiv \mathfrak{g}_{2,2}$ in the orthogonal decomposition:

$$\mathbb{U} = \mathbb{H} \oplus \mathbb{K} \quad ; \quad \begin{cases} [\mathbb{H}, \mathbb{H}] \subset \mathbb{H} \\ [\mathbb{H}, \mathbb{K}] \subset \mathbb{K} \\ [\mathbb{K}, \mathbb{K}] \subset \mathbb{H} \end{cases} \quad (6.29)$$

Since our quaternionic Kähler manifold is a symmetric space it follows that Lie algebra \mathbb{H} must be isomorphic with the holonomy algebra $\text{Hol} = \mathfrak{su}_I(2) \oplus \mathfrak{su}_{II}(2)$ that we have calculated in the previous subsection. By definition the Lie algebra \mathbb{H} is the maximal compact subalgebra which for maximal split algebras has a universal definition in terms of the step operators associated with the positive roots E^α and their conjugates $E^{-\alpha}$. In the case of $\mathfrak{g}_{2,2}$ which has six positive roots we can write:

$$\mathbb{H} \equiv \text{span}_{\mathbb{R}} \{E^{\alpha_1} - E^{\alpha_1}, E^{\alpha_2} - E^{\alpha_2}, E^{\alpha_3} - E^{\alpha_3}, E^{\alpha_4} - E^{\alpha_4}, E^{\alpha_5} - E^{\alpha_5}, E^{\alpha_6} - E^{\alpha_6}\} \quad (6.30)$$

The structure of (6.30) is the following:

$$\mathbb{H} = \mathfrak{su}_I(2) \oplus \mathfrak{su}_{II}(2) \quad (6.31)$$

where the generators of the two subalgebras are:

$$j_{[I]}^x = \begin{pmatrix} \frac{-3E^{-\alpha_1} + 3E^{\alpha_1} + \sqrt{3}(E^{\alpha_6} - E^{-\alpha_6})}{6\sqrt{2}} \\ \frac{3E^{-\alpha_3} - 3E^{\alpha_3} + \sqrt{3}(E^{-\alpha_5} - E^{\alpha_5})}{6\sqrt{2}} \\ \frac{\sqrt{3}(E^{\alpha_2} - E^{-\alpha_2}) + 3(E^{-\alpha_4} - E^{\alpha_4})}{6\sqrt{2}} \end{pmatrix} \quad (6.32)$$

and

$$j_{[II]}^x = \begin{pmatrix} \frac{-E^{-\alpha_1} + E^{\alpha_1} + \sqrt{3}(E^{-\alpha_6} - E^{\alpha_6})}{2\sqrt{2}} \\ \frac{-E^{-\alpha_3} + E^{\alpha_3} + \sqrt{3}(E^{-\alpha_5} - E^{\alpha_5})}{2\sqrt{2}} \\ \frac{\sqrt{3}(E^{-\alpha_2} - E^{\alpha_2}) + E^{-\alpha_4} - E^{\alpha_4}}{2\sqrt{2}} \end{pmatrix} \quad (6.33)$$

and satisfy among themselves the same relations (6.23) as their homologous generators $J_{[I]}^x$ and $J_{[II]}^x$. In eq.(6.31) we have used the same notation as in eq.(6.28) using the obligatory homomorphism between the the holonomy algebra Hol and the isotropy subalgebra \mathbb{H} . The precise correspondence between generators of one algebra and generators of the other will be establishe in the next subsection by means of the use of the coset representative.

6.7 THE COSET REPRESENTATIVE

The next step in the development of the coset approach is the construction of the solvable coset representative $\mathbb{L}_{Solv}(\phi)$, advocated in eq.s(4.65-4.66), namely a coordinate dependent element of the Borel group of $\mathfrak{g}_{(2,2)}$ such that the Maurer Cartan form

$$\Xi = \mathbb{L}_{Solv}(\phi)^{-1} d\mathbb{L}_{Solv}(\phi) \quad (6.34)$$

projected along the Borel algebra generators, as given in eq.(6.18), reproduces the vielbein of eq.(6.13). The appropriate coset representative is obtained by exponentiating the Borel Lie algebra and the precise recipe is provided below. First define:

$$\begin{aligned} \mathbf{L}_0^E &= \frac{1}{\sqrt{3}} \mathcal{H}_2 \quad ; \quad \mathbf{L}_+^E = -\sqrt{\frac{2}{3}} E^{\alpha_6} \\ \mathbf{L}_0 &= \mathcal{H}_1 \quad ; \quad \mathbf{L}_+ = \sqrt{2} E^{\alpha_1} \\ \mathbf{W}^I &= \sqrt{\frac{2}{3}} \{E^{\alpha_4}, E^{\alpha_5}, -E^{\alpha_3}, -E^{\alpha_2}, \} \end{aligned} \quad (6.35)$$

and then set:

$$\begin{aligned} \mathbb{L} = & \exp[a\mathbf{L}_+^E] \cdot \exp\left[\sqrt{2}(Z_1\mathbf{W}^1 + Z_3\mathbf{W}^3)\right] \cdot \\ & \cdot \exp\left[\sqrt{2}(Z_1\mathbf{W}^1 + Z_3\mathbf{W}^3)\right] \cdot \exp[y\mathbf{L}_+] \cdot \exp[h\mathbf{L}_0] \cdot \exp[U\mathbf{L}_0^E] \end{aligned} \quad (6.36)$$

By explicit evaluation we obtain the result displayed in the appendix in formulae (A.1) and (A.2) and we verify that, if we set:

$$\mathfrak{T}_I = \{\mathbf{L}_0^E, \mathbf{L}_0, \mathbf{L}_+^E, \mathbf{L}_+, \mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4\} \quad (6.37)$$

upon substitution of (A.2) into the Maurer Cartan form (6.34) we obtain:

$$\mathbb{L}_{Solv}(\phi)^{-1} d\mathbb{L}_{Solv}(\phi) = \sum_{I=1}^8 \mathfrak{T}_I E_{\mathcal{QM}}^I \quad (6.38)$$

the forms $E_{\mathcal{QM}}^I$ being given in equation (6.13). Alternatively we can also write:

$$\mathbb{L}_{Solv}(\phi)^{-1} d\mathbb{L}_{Solv}(\phi) = \sum_{x=1}^3 \left(\omega_x^{[I]} j_{[I]}^x \oplus \omega_x^{[II]} j_{[II]}^x \right) \oplus \sum_I^8 \mathbf{T}_I E_{SQ}^I \quad (6.39)$$

In the above equation $\omega_x^{[I]}$ and $\omega_x^{[II]}$ are the components of the spin connections given in eq. (6.27), $j_{[I]}^x$ and $j_{[II]}^x$ are the generators of \mathbb{H} defined in eq. (6.32,6.33) and \mathbf{T}_I denotes a suitable base of generators in the \mathbb{K} subspace of $\mathfrak{g}_{(2,2)}$ defined as:

$$\begin{aligned} \mathbb{K} \equiv & , \text{span}_{\mathbb{R}} \{ \mathcal{H}_1, \mathcal{H}_2, E^{\alpha_1} + E^{\alpha_1}, E^{\alpha_2} + E^{\alpha_2}, E^{\alpha_3} + E^{\alpha_3} \\ & E^{\alpha_4} + E^{\alpha_4}, E^{\alpha_5} + E^{\alpha_5}, E^{\alpha_6} + E^{\alpha_6} \} \end{aligned} \quad (6.40)$$

The precise form of the generators \mathbf{T}_I is not relevant to our purposes and we omit it. The key point is instead the identification of the generators $j_{[I]}^x$ of \mathbb{H} with generators $J_{[I]}^x$ of the holonomy algebra. This provides us with the knowledge of the quaternionic complex structures within the algebra $\mathbb{U}_{\mathcal{Q}}$ and allows to calculate the tri-holomorphic moment map of any generator $\mathbf{t} \in \mathbb{U}_{\mathcal{Q}}$ by means of the formula (4.62) which in our case reads:

$$\mathcal{P}_{\mathbf{t}}^x = \frac{1}{2} \text{Tr}_{\mathbf{7}} \left(j_{[I]}^x \mathbb{L}_{Solv}^{-1} \mathbf{t} \mathbb{L}_{Solv} \right) \quad (6.41)$$

having denoted by $\mathbf{7}$ the 7-dimensional fundamental representation of $\mathfrak{g}_{(2,2)}$.

The Starobinsky potential As an immediate application of eq.(6.41) one can retrieve the results of [1] on the inclusion of the Starobinsky potential into supergravity. In section 5.2 we presented a general discussion of the gaugings of nilpotent generators in the Special Kähler subalgebra $\mathbb{U}_{SK} \subset \mathbb{U}_{\mathcal{Q}}$. In the present case where $\mathbb{U}_{SK} = \mathfrak{sl}(2, \mathbb{R})$ the only available nilpotent operator is L_+ and from the general formula (2.55) applied to the case where the metric is given by (6.3) and the complex coordinate is parameterized as in eq.(6.4) we find:

$$\mathcal{P}_{L_+} = \text{const} \times \exp[-h] = \text{const} \times (\text{Im } z)^{-1} \quad (6.42)$$

This result inserted into the general formula (5.19) yields

$$V(h) = \text{const} \times (\exp[-h] + \kappa)^2 \quad (6.43)$$

which is indeed the Starobinsky potential, since, once expressed in terms of h , the Kähler potential is exactly $\mathcal{K} = 3h$. The same result is directly obtained with precise coefficients by inserting in eq.(6.41) the 7-dimensional image of L_+ in the fundamental representation of $\mathfrak{g}_{(2,2)}$.

7 THE $\mathrm{Sp}(6, \mathbb{R})/\mathrm{SU}(3) \times \mathrm{U}(1)$ - MODEL AND ITS C-MAP IMAGE.

Next we consider the Special Kähler manifold

$$\mathcal{M}_{\mathrm{Sp}6} = \frac{\mathrm{Sp}(6, \mathbb{R})}{\mathrm{SU}(3) \times \mathrm{U}(1)} \quad (7.1)$$

and its c -map image which is the following quaternionic manifold:

$$c\text{-map} : \mathcal{M}_{\mathrm{Sp}6} \mapsto \mathcal{QM}_{F4} \equiv \frac{\mathrm{F}_{(4,4)}}{\mathrm{SU}(2) \times \mathrm{USp}(6)} \quad (7.2)$$

$\mathcal{M}_{\mathrm{Sp}6}$ belongs to the magic square of exceptional special Kähler manifolds whose quaternionic c -map is a homogeneous symmetric space having, as it is evident from (7.2), an exceptional Lie group as isometry group.

We begin by illustrating some general properties of this remarkable manifold. First of all, in order to discuss them adequately we need to choose a basis for the $\mathfrak{sp}(6, \mathbb{R})$ Lie algebra. Since we are not interested in solving Lax equations we do not choose the basis where the matrices of the Borel subalgebra are upper triangular. We rather use the basis where the symplectic preserved metric is the standard one, namely:

$$\mathbb{C} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \quad (7.3)$$

This traditional choice allows to describe in a simple way other aspects of the manifold geometry that are more relevant to our present purposes.

According to the above choice, an element of the $\mathrm{Sp}(6, \mathbb{R})$ group and an element of the $\mathfrak{sp}(6, \mathbb{R})$ Lie-algebra are matrices respectively fulfilling the following two constraints:

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)^T \mathbb{C} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) = \mathbb{C} \quad ; \quad \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right)^T \mathbb{C} + \mathbb{C} \left(\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right) = 0 \quad (7.4)$$

where A, B, C, D , $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are 3×3 blocks. By means of the so called Cayley transformation

$$\mathcal{C} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix} \quad (7.5)$$

a real element of the symplectic group (or algebra) can be mapped into a matrix that is simultaneously symplectic and pseudounitary:

$$\mathcal{S} = \mathcal{C}^\dagger \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \mathcal{C} = \begin{pmatrix} U_0 & U_1^* \\ U_1 & U_0^* \end{pmatrix} \in \text{Sp}(6, \mathbb{C}) \cap \text{SU}(3, 3) \quad (7.6)$$

The diagonal blocks $U_0 \in \text{U}(3)$ span the H-subgroup of the coset (7.1). This allows to introduce a set of projective coordinates that parameterize the points of the manifold (7.1) and have a nice fractional linear transformation under the action of the group $\text{Sp}(6, \mathbb{R})$. Given any coset parameterization

$$\left(\begin{array}{c|c} A(\phi) & B(\phi) \\ \hline C(\phi) & D(\phi) \end{array} \right) \in \text{Sp}(6, \mathbb{R}) \quad (7.7)$$

namely a family of symplectic group elements depending on 12 parameters ϕ^i such that each different choice of the ϕ^i provides a representative of a different equivalence class in (7.1), we can construct the following, *symmetric complex matrix*:

$$Z(\phi) \equiv (A(\phi) - iB(\phi)) (C(\phi) - iD(\phi))^{-1} \quad (7.8)$$

which has a very simple transformation under the action of the symplectic group. Let us consider the action of any element of $\text{Sp}(6, \mathbb{R})$ on the coset representative. We have:

$$\underbrace{\left(\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array} \right)}_{= \mathfrak{g} \in \text{Sp}(6, \mathbb{R})} \left(\begin{array}{c|c} A(\phi) & B(\phi) \\ \hline C(\phi) & D(\phi) \end{array} \right) = \left(\begin{array}{c|c} A(\phi') & B(\phi') \\ \hline C(\phi') & D(\phi') \end{array} \right) \text{H}(\phi, \mathfrak{g}) \quad (7.9)$$

where ϕ' is the label of a new equivalence class and $\text{H}(\phi, \mathfrak{g}) \in \text{U}(3)$ is a suitable H-compensator. Calculating the matrix $Z(\phi')$ according to the definition (7.8) we find that it is related to $Z(\phi)$ by a simple linear fractional transformation (generalized to matrices):

$$Z(\phi') = (AZ(\phi) + B) (CZ(\phi) + D)^{-1} \quad (7.10)$$

Formula (7.10) is of crucial relevance and requires several comments. From a mathematical point of view, (7.10) is the well known generalization of the action of the $\text{SL}(2, \mathbb{R}) \simeq \text{Sp}(2, \mathbb{R})$ group on the upper complex plane of Poincaré-Lobachevsky. The complex numbers z with positive imaginary parts ($\text{Im}z > 0$) are replaced by the complex symmetric matrices Z_{ij} whose imaginary part is positive definite. Such matrices constitute the so named **upper Siegel plane**, which indeed is homeomorphic to the coset $\text{Sp}(2n, \mathbb{R})/\text{U}(n)$. From the physical point of view (7.10) is just identical to the Gaillard-Zumino formula for the construction of the kinetic matrix $\mathcal{N}_{\Lambda\Sigma}$ which appears in the lagrangian of the vector fields in $\mathcal{N} = 2$ supergravity and is rooted in the structure of special Kähler geometry. Indeed for any special Kähler manifold \mathcal{M}_n of complex dimension n that is also a symmetric space G/H , there exists a so named **W**-representation of G , which is symplectic, has dimension $2n + 2$ and hosts the electric and magnetic field strengths of the model. Such a representation defines a symplectic embedding:

$$\text{G} \rightarrow \text{Sp}(2n + 2, \mathbb{R}) \quad (7.11)$$

which associates to any coset representative $\mathfrak{g}(\phi) \in \text{G}/\text{H}$ its corresponding symplectic $(2n + 2) \times (2n + 2)$ representation $\left(\begin{array}{c|c} A(\phi) & B(\phi) \\ \hline C(\phi) & D(\phi) \end{array} \right)$. From this latter, utilizing the recipe provided by formula (7.10) we obtain

an $(n+1) \times (n+1)$ complex symmetric matrix to be identified with the appropriate \mathcal{N} kinetic matrix largely discussed and utilized in section 2.4.

The peculiarity of the $\mathcal{N} = 2$ model under investigation is that the original isometry group G is already symplectic so that we can utilize the Gaillard-Zumino formula (7.10) in the fundamental 6 dimensional representation in order to construct a Siegel parametrization of the coset in terms of a symmetric complex 3×3 matrix Z . The \mathbf{W} -representation is the $\mathbf{14}'$ and this defines the embedding:

$$\mathrm{Sp}(6, \mathbb{R}) \mapsto \mathrm{Sp}(14, \mathbb{R}) \quad (7.12)$$

from which we can construct the 7×7 kinetic matrix $\mathcal{N}(Z)$.

THE TRANSITIVE ACTION OF $\mathrm{Sp}(6, \mathbb{R})$ ON THE UPPER SIEGEL PLANE. Before proceeding with the actual construction of the Lie algebra let us comment on the transitive action of the symplectic group on the Siegel plane. Focusing on the the formula (7.10), consider the $\mathrm{Sp}(6, \mathbb{R})$ parabolic subgroup composed by the following matrices:

$$\mathfrak{g}(B) = \left(\begin{array}{c|c} \mathbf{1}_{3 \times 3} & B \\ \hline \mathbf{0}_{3 \times 3} & \mathbf{1}_{3 \times 3} \end{array} \right) \quad (7.13)$$

where B is symmetric and real. By means of such a subgroup we can always map a generic Z matrix into one that has vanishing real part $\mathrm{Re}Z = 0$. Next consider the action on the residual imaginary part of Z of the $\mathrm{GL}(3, \mathbb{R}) \subset \mathrm{Sp}(6, \mathbb{R})$ subgroup composed by the matrices:

$$\mathfrak{g}(B) = \left(\begin{array}{c|c} \mathcal{A} & \mathbf{0}_{3 \times 3} \\ \hline \mathbf{0}_{3 \times 3} & (\mathcal{A}^T)^{-1} \end{array} \right) \quad ; \quad \mathcal{A} \in \mathrm{GL}(3, \mathbb{R}) \quad (7.14)$$

We obtain:

$$\mathrm{Im}Z \mapsto \mathcal{A} \mathrm{Im}Z \mathcal{A}^T \quad (7.15)$$

Choosing $\mathcal{A} = (\mathrm{Im}Z)^{\frac{1}{2}}$, which is always possible since $\mathrm{Im}Z$ is positive definite we can reduce the imaginary part to the identity matrix. This shows the transitive action of the symplectic group on the Siegel plane and also provides a nice coset parameterization of the coset manifold. Indeed we can introduce the following matrix:

$$\mathfrak{g}(Z) \equiv \left(\begin{array}{c|c} (\mathrm{Im}Z)^{\frac{1}{2}} & \mathrm{Re}Z (\mathrm{Im}Z)^{-\frac{1}{2}} \\ \hline \mathbf{0} & (\mathrm{Im}Z)^{-\frac{1}{2}} \end{array} \right) \quad (7.16)$$

which maps the origin of the manifold $i\mathbf{1}_{3 \times 3}$ in the complex symmetric matrix Z .

7.1 THE $\mathfrak{sp}(6, \mathbb{R})$ LIE ALGEBRA

From the point of view of the Dynkin classification the Lie algebra $\mathfrak{sp}(6, \mathbb{R})$ is the maximally split real section of the complex Lie algebra C_3 whose Dynkin diagram is displayed in fig.1. The root system is composed of

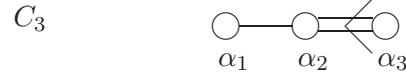


Figure 1: The Dynkin diagram of C_3 .

18-roots whose subset of 9 positive ones is displayed here below:

$$\begin{array}{c} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \\ \alpha_9 \end{array} = \begin{array}{cc} \alpha_1 & \{1, -1, 0\} \\ \alpha_2 & \{0, 1, -1\} \\ \alpha_3 & \{0, 0, 2\} \\ \alpha_1 + \alpha_2 & \{1, 0, -1\} \\ \alpha_2 + \alpha_3 & \{0, 1, 1\} \\ \alpha_1 + \alpha_2 + \alpha_3 & \{1, 0, 1\} \\ 2\alpha_2 + \alpha_3 & \{0, 2, 0\} \\ \alpha_1 + 2\alpha_2 + \alpha_3 & \{1, 1, 0\} \\ 2\alpha_1 + 2\alpha_2 + \alpha_3 & \{2, 0, 0\} \end{array} \quad (7.17)$$

The simple roots are the first three. Of the remaining 6 we have provided both their expression in terms of the simple roots and their realization as three-vectors in \mathbb{R}^3 . Such a realization is spelled out also for the simple roots. Next we present the basis of 6×6 matrices that fulfill the standard commutation relations of the Lie Algebra in the Cartan Weyl basis.

Cartan Generators. The Cartan generators are named \mathcal{H}^i and can be easily read-off from the following formula:

$$\sum_{i=1}^3 h_i \mathcal{H}^i = \begin{pmatrix} h_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & h_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & -h_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -h_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -h_3 \end{pmatrix} \quad (7.18)$$

by collecting the coefficient of the parameter h_i .

POSITIVE ROOT STEP OPERATORS. The step operator associated with the positive root α_i is named \mathcal{E}^{α_i} and can be easily read-off from the following formula:

$$\sum_{i=1}^9 a_i \mathcal{E}^{\alpha_i} = \begin{pmatrix} 0 & a_1 & a_4 & \sqrt{2}a_9 & a_8 & a_6 \\ 0 & 0 & a_2 & a_8 & \sqrt{2}a_7 & a_5 \\ 0 & 0 & 0 & a_6 & a_5 & \sqrt{2}a_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & -a_4 & -a_2 & 0 \end{pmatrix} \quad (7.19)$$

by collecting the coefficient of the parameter a_i .

NEGATIVE ROOT STEP OPERATORS. The step operator associated with the negative root $-\alpha_i$ is named $\mathcal{E}^{-\alpha_i}$ and can be easily read-off from the following formula:

$$\sum_{i=1}^9 b_i \mathcal{E}^{-\alpha_i} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 \\ b_4 & b_2 & 0 & 0 & 0 & 0 \\ \sqrt{2}b_9 & b_8 & b_6 & 0 & -b_1 & -b_4 \\ b_8 & \sqrt{2}b_7 & b_5 & 0 & 0 & -b_2 \\ b_6 & b_5 & \sqrt{2}b_3 & 0 & 0 & 0 \end{pmatrix} \quad (7.20)$$

by collecting the coefficient of the parameter b_i .

7.2 THE REPRESENTATION $14'$

The $14'$ representation of $\mathfrak{sp}(6, \mathbb{R})$ which plays the role \mathbf{W} -representation for the special manifold under consideration is defined as the representation obeyed by the three-times antisymmetric tensors with vanishing \mathbb{C} -traces, namely:

$$\underbrace{t_{ABC}}_{\text{antisymmetric in } A, B, C} \times \mathbb{C}^{BC} = 0 \quad (7.21)$$

The generators are constructed in the appendix, subsection A.2, and displayed in eq.s (A.11), (A.12) and (A.13).

7.3 THE HOLOMORPHIC SYMPLECTIC SECTION AND ITS TRANSFORMATION IN THE $14'$

In order to construct the special geometry of the manifold (7.1) we need to introduce the holomorphic symplectic section that, by definition, should transform in the $14'$ representation of $\text{Sp}(6, \mathbb{R})$. To this effect, we choose as special coordinates the components of the symmetric complex matrix defined by eq.(7.10) and we choose a lexicographic order to enumerate its independent components, namely we set:

$$Z = \begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} \quad (7.22)$$

Next we introduce the holomorphic prepotential defined by:

$$\begin{aligned}\mathcal{F} &\equiv Z_{a,i} Z_{b,j} Z_{c,k} \epsilon^{abc} \epsilon^{ijk} \\ &= -6 (z_6 z_2^2 - 2z_3 z_5 z_2 + z_3^2 z_4 + z_1 (z_5^2 - z_4 z_6))\end{aligned}\quad (7.23)$$

and we can introduce a first ansatz for the symplectic section by writing:

$$\begin{aligned}\tilde{\Omega} &= \left\{ 1, z^I, \mathcal{F}, \frac{\partial \mathcal{F}}{\partial z^J} \right\} \\ &= \left\{ 1, z_1, z_2, z_3, z_4, z_5, z_6, -6 (z_6 z_2^2 - 2z_3 z_5 z_2 + z_3^2 z_4 + z_1 (z_5^2 - z_4 z_6)), \right. \\ &\quad \left. 6z_4 z_6 - 6z_5^2, 12 (z_3 z_5 - z_2 z_6), 12 (z_2 z_5 - z_3 z_4), 6z_1 z_6 - 6z_3^2, 12 (z_2 z_3 - z_1 z_5), 6z_1 z_4 - 6z_2^2 \right\}\end{aligned}\quad (7.24)$$

In order to match the transformation of this holomorphic section with the transformations of the $\mathbf{14}'$ representation as we defined it in subsection 7.3 we still need a change of basis. Consider the following matrix

$$\mathfrak{S} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{3\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}\quad (7.25)$$

and define:

$$\Omega(Z) = \mathfrak{S} \tilde{\Omega}(Z) = \begin{pmatrix} \sqrt{2}z_6 \\ \sqrt{2}(z_1z_6 - z_3^2) \\ \sqrt{2}(z_5^2 - z_4z_6) \\ -\sqrt{2}(z_6z_2^2 - 2z_3z_5z_2 + z_3^2z_4 + z_1(z_5^2 - z_4z_6)) \\ 2z_2z_3 - 2z_1z_5 \\ 2z_3z_4 - 2z_2z_5 \\ 2z_3z_5 - 2z_2z_6 \\ \sqrt{2}(z_1z_4 - z_2^2) \\ -\sqrt{2}z_4 \\ \sqrt{2}z_1 \\ \sqrt{2} \\ -2z_5 \\ 2z_3 \\ -2z_2 \end{pmatrix} \quad (7.26)$$

Naming $\mathcal{D}_{14}[g]$ the 14-dimensional representation of a finite element $g, \in \text{Sp}(6, \mathbb{R})$ of the symplectic group that corresponds to the representation of the algebra as we constructed it above, the holomorphic symplectic section (7.26) transforms in the following way:

$$\Omega[(AZ + B)(CZ + D)^{-1}] = \frac{1}{\text{Det}(CZ + D)} \mathcal{D}_{14} \left[\left(\frac{A}{C} \middle| \frac{B}{D} \right) \right] \Omega[Z] \quad (7.27)$$

The formula (7.27) can be in particular applied to the case where the original Z is the origin of the coset manifold: $Z_0 = i\mathbf{1}_{3 \times 3}$. In that case, recalling eq. (7.16) we find:

$$\Omega[Z_0] = \{i\sqrt{2}, -\sqrt{2}, \sqrt{2}, -i\sqrt{2}, 0, 0, 0, -\sqrt{2}, -i\sqrt{2}, i\sqrt{2}, \sqrt{2}, 0, 0, 0\} \quad (7.28)$$

and

$$\Omega[Z_0] = \sqrt{\text{Det}[\text{Im}Z]} \times \mathcal{D}_{14} \left[\left(\frac{(\text{Im}Z)^{\frac{1}{2}}}{\mathbf{0}} \middle| \frac{\text{Re}Z (\text{Im}Z)^{-\frac{1}{2}}}{(\text{Im}Z)^{-\frac{1}{2}}} \right) \right] \cdot \Omega[Z_0] \quad (7.29)$$

7.4 THE KÄHLER POTENTIAL AND THE METRIC

Provided with this information we can now write the explicit form of the Kähler potential and of the Kähler metric for the manifold (7.1) according to the rules of special Kähler geometry. We have:

$$\begin{aligned} \mathcal{K} &\equiv -\log(i\Omega[Z] \mathbb{C}_{14} \overline{\Omega}[\bar{Z}]) \\ &= -\log(2i(-z_6z_2^2 + \bar{z}_6z_2^2 + 2z_6\bar{z}_2z_2 - 2z_5\bar{z}_3z_2 + 2\bar{z}_3\bar{z}_5z_2 - 2\bar{z}_2\bar{z}_6z_2 - z_6\bar{z}_2^2 - z_4\bar{z}_3^2 + \bar{z}_1\bar{z}_5^2 + \\ &\quad z_5^2\bar{z}_1 - z_4z_6\bar{z}_1 + 2z_5\bar{z}_2\bar{z}_3 + \bar{z}_3^2\bar{z}_4 + z_6\bar{z}_1\bar{z}_4 + z_3^2(\bar{z}_4 - z_4) - 2z_5\bar{z}_1\bar{z}_5 - 2\bar{z}_2\bar{z}_3\bar{z}_5 \\ &\quad + 2z_3(-z_5\bar{z}_2 + \bar{z}_5z_2 + z_4\bar{z}_3 - \bar{z}_3\bar{z}_4 + z_2(z_5 - \bar{z}_5)) + \bar{z}_2^2\bar{z}_6 + z_4\bar{z}_1\bar{z}_6 - \bar{z}_1\bar{z}_4\bar{z}_6 \\ &\quad - z_1(z_5^2 - 2\bar{z}_5z_5 + \bar{z}_5^2 + z_6\bar{z}_4 - \bar{z}_4\bar{z}_6 + z_4(\bar{z}_6 - z_6)))) \end{aligned} \quad (7.30)$$

and the line element on the manifold, in terms of the special coordinates z_i takes the standard form:

$$ds_K^2 = \frac{\partial}{\partial z^i} \frac{\partial}{\partial \bar{z}^j} \mathcal{K} dz^i \otimes d\bar{z}^j \quad (7.31)$$

The explicit form of ds_K^2 in terms of the special coordinate z^i can be worked out by simple derivatives, yet its explicit form is quite lengthy and so much involved that we think it better not to display it. For the purposes that we pursue we rather prefer to write the form of the metric in terms of solvable real coordinates.

7.4.1 THE SOLVABLE PARAMETRIZATION

The transition to a solvable parametrization of the coset is rather simple. Let us define the solvable coset representative as the product of the exponentials of all the generators of the Borel subalgebra of $\mathfrak{sp}(6, \mathbb{R})$:

$$\mathbb{L}(h, p) = \prod_{i=1}^9 \exp [p_{10-i} \mathcal{E}^{\alpha_{10-i}}] \prod_{j=3}^3 \exp [h_j \mathcal{H}^j] =$$

e^{h_1}	$e^{h_2 p_1}$	$e^{h_3 p_4}$	$e^{-h_1} \left(\sqrt{2} p_1 p_2 p_3 p_4 + (p_1 p_2 - p_4) p_6 - p_1 p_8 + \sqrt{2} p_9 \right)$	$e^{-h_2} \left(-\sqrt{2} p_2 p_3 p_4 - p_2 p_6 + p_8 \right)$	$e^{-h_3} \left(\sqrt{2} p_3 p_4 + p_6 \right)$
0	e^{h_2}	$e^{h_3 p_2}$	$e^{-h_1} \left((p_1 p_2 - p_4) p_5 - \sqrt{2} p_1 p_7 + p_8 \right)$	$e^{-h_2} \left(\sqrt{2} p_7 - p_2 p_5 \right)$	$e^{-h_3} p_5$
0	0	e^{h_3}	$e^{-h_1} \left(\sqrt{2} p_1 p_2 p_3 - p_1 p_5 + p_6 \right)$	$e^{-h_2} \left(p_5 - \sqrt{2} p_2 p_3 \right)$	$\sqrt{2} e^{-h_3} p_3$
0	0	0	e^{-h_1}	0	0
0	0	0	$-e^{-h_1} p_1$	e^{-h_2}	0
0	0	0	$e^{-h_1} (p_1 p_2 - p_4)$	$-e^{-h_2} p_2$	e^{-h_3}

(7.32)

The real coordinates of the manifold are now the 12 parameters:

$$\text{coordinates} \equiv \{h_1, \dots, h_3, p_1, \dots, p_9\} \quad (7.33)$$

Extracting the complex matrix Z from the symplectic matrix $\mathbb{L}(h, p)$ we find:

$$Z(h, p) =$$

$ie^{2h_2} p_1^2 + ie^{2h_1} + (\sqrt{2} p_3 + ie^{2h_3}) p_4^2 + \sqrt{2} p_9$	$ie^{2h_2} p_1 + ie^{2h_3} p_2 p_4 + p_8$	$(\sqrt{2} p_3 + ie^{2h_3}) p_4 + p_6$
$ie^{2h_2} p_1 + ie^{2h_3} p_2 p_4 + p_8$	$ie^{2h_3} p_2^2 + ie^{2h_2} + \sqrt{2} p_7$	$ie^{2h_3} p_2 + p_5$
$(\sqrt{2} p_3 + ie^{2h_3}) p_4 + p_6$	$ie^{2h_3} p_2 + p_5$	$\sqrt{2} p_3 + ie^{2h_3}$

(7.34)

which defines the coordinate transformation from the special to the solvable coordinates:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} \sqrt{2} p_3 p_4^2 + i (e^{2h_2} p_1^2 + e^{2h_1} + e^{2h_3} p_4^2) + \sqrt{2} p_9 \\ i (e^{2h_2} p_1 + e^{2h_3} p_2 p_4) + p_8 \\ ie^{2h_3} p_4 + \sqrt{2} p_3 p_4 + p_6 \\ i (e^{2h_3} p_2^2 + e^{2h_2}) + \sqrt{2} p_7 \\ ie^{2h_3} p_2 + p_5 \\ \sqrt{2} p_3 + ie^{2h_3} \end{pmatrix} \quad (7.35)$$

Inserting such a coordinate transformation into the Kähler metric (7.31) we obtain its form in terms of the real coordinates (7.33). For the explicit form of the metric, we refer the reader to the appendix, eq. (A.14). The complete metric is quite formidable (A.14) since it contains a total of 100 terms. It has however quite simple properties when we sit in the neighborhood of the coset origin, in particular at $p_i \sim 0$. In this case it drastically simplifies and becomes diagonal:

$$ds_K^2 \xrightarrow{p_i \rightarrow 0} dh_1^2 + dh_2^2 + dh_3^2 + \frac{1}{2}e^{2h_2-2h_1}dp_1^2 + \frac{1}{2}e^{2h_3-2h_2}dp_2^2 + \frac{1}{2}e^{-4h_3}dp_3^2 + \frac{1}{2}e^{2h_3-2h_1}dp_4^2 \\ + \frac{1}{2}e^{-2h_2-2h_3}dp_5^2 + \frac{1}{2}e^{-2h_1-2h_3}dp_6^2 + \frac{1}{2}e^{-4h_2}dp_7^2 + \frac{1}{2}e^{-2h_1-2h_2}dp_8^2 + \frac{1}{2}e^{-4h_1}dp_9^2 \quad (7.36)$$

which shows that it is positive definite as it should be. It is also interesting to note that if the truncation to the Cartan is permitted by the potential, then we just have three dilatons with canonical kinetic terms.

7.5 THE QUARTIC INVARIANT IN THE 14'

Of crucial relevance for the analysis of Black Hole charges and in general for the classification of orbits in the \mathbf{W} -representation is the quartic symplectic invariant. Given a 14-vector

$$\mathcal{Q} = \{q_1, q_2 \dots, q_{14}\} \quad (7.37)$$

the standard form of this invariant can be expressed in the following manifestly $\text{Sp}(6, \mathbb{R})$ -invariant form (see for instance [76])

$$\mathfrak{J}_4(\mathcal{Q}) = -\frac{n_V(2n_V+1)}{6d} (\Lambda_a)_{\alpha\beta} (\Lambda^a)_{\gamma\delta} \mathcal{Q}^\alpha \mathcal{Q}^\beta \mathcal{Q}^\gamma \mathcal{Q}^\delta, \quad (7.38)$$

where in our case $n_V = 7$ and $d = \dim \text{Sp}(6, \mathbb{R}) = 21$, the symplectic indices are raised and lowered by $\mathbb{C}_1 4^{\alpha\beta}$ and $\mathbb{C}_{14\alpha\beta}$ and the index a is raised by the inverse of $\eta_{ab} \equiv \text{Tr}(\Lambda_a \Lambda_b)$. The explicit form of $\mathfrak{J}_4(\mathcal{Q})$ reads:

$$\begin{aligned} \mathfrak{J}_4(\mathcal{Q}) = & -2q_1q_9q_5^2 + 2q_3q_{11}q_5^2 - 2\sqrt{2}q_6q_7q_{11}q_5 - 2q_1q_8q_{12}q_5 + 2q_2q_9q_{12}q_5 - 2q_3q_{10}q_{12}q_5 \\ & + 2q_4q_{11}q_{12}q_5 - 2\sqrt{2}q_7q_9q_{13}q_5 + 2\sqrt{2}q_1q_6q_{14}q_5 + 2\sqrt{2}q_3q_{13}q_{14}q_5 + q_1^2q_8^2 \\ & + q_2^2q_9^2 + q_3^2q_{10}^2 + q_4^2q_{11}^2 + 2q_2q_8q_{12}^2 - 2q_4q_{10}q_{12}^2 - 2q_3q_8q_{13}^2 + 2q_4q_9q_{13}^2 - 2q_2q_3q_{14}^2 \\ & - 2q_1q_4q_{14}^2 + 2q_1q_2q_8q_9 + 2q_1q_6^2q_{10} + 2q_1q_3q_8q_{10} - 2q_7^2q_9q_{10} - 2q_2q_3q_9q_{10} \\ & - 4q_1q_4q_9q_{10} - 2q_2q_6^2q_{11} - 2q_7^2q_8q_{11} - 4q_2q_3q_8q_{11} \\ & - 2q_1q_4q_8q_{11} + 2q_2q_4q_9q_{11} + 2q_3q_4q_{10}q_{11} \\ & + 2\sqrt{2}q_6q_7q_{10}q_{12} - 2q_1q_6q_8q_{13} - 2q_2q_6q_9q_{13} + 2q_3q_6q_{10}q_{13} \\ & + 2q_4q_6q_{11}q_{13} - 2\sqrt{2}q_7q_8q_{12}q_{13} + 2q_1q_7q_8q_{14} \\ & + 2q_2q_7q_9q_{14} + 2q_3q_7q_{10}q_{14} + 2q_4q_7q_{11}q_{14} - 2\sqrt{2}q_2q_6q_{12}q_{14} + 2\sqrt{2}q_4q_{12}q_{13}q_{14} \end{aligned} \quad (7.39)$$

7.6 TRUNCATION TO THE STU -MODEL

Next we analyze how the STU -model is embedded into the $\text{Sp}(6, \mathbb{R})$ -model. At the level of the special coordinates the truncation to the STU -model is very simply done. It suffices to set to zero the complex

coordinates z_2, z_3, z_5 keeping only z_1, z_4, z_6 that can be identified with the fields S, T, U . When we do so the symplectic section reduces as follows:

$$\Omega \left[Z \begin{pmatrix} z_1 & 0 & 0 \\ 0 & z_4 & 0 \\ 0 & 0 & z_6 \end{pmatrix} \right] = \begin{pmatrix} \sqrt{2}z_6 \\ \sqrt{2}z_1z_6 \\ -\sqrt{2}z_4z_6 \\ \sqrt{2}z_1z_4z_6 \\ 0 \\ 0 \\ 0 \\ \sqrt{2}z_1z_4 \\ -\sqrt{2}z_4 \\ \sqrt{2}z_1 \\ \sqrt{2} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad (7.40)$$

and the Kähler potential reduces to:

$$\mathcal{K} \rightarrow -\log [2i (z_1 - \bar{z}_1) (z_4 - \bar{z}_4) (z_6 - \bar{z}_6)] \quad (7.41)$$

which yields three copies of the Poincaré metric, one for each of the three $\frac{\text{SL}(2, \mathbb{R})}{\text{SO}(2)}$ submanifolds.

The result (7.40) is in agreement with the decomposition of the $\mathbf{14}'$ of $\mathfrak{sp}(6, \mathbb{R})$ with respect to the three subalgebras $\mathfrak{sl}(2)$:

$$\mathbf{14}' \xrightarrow{\mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2)} (\mathbf{2}, \mathbf{2}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}) \quad (7.42)$$

From (7.40) we also learn that the directions $\{1, 2, 3, 4, 8, 9, 10, 11\}$ of the $\mathbf{14}'$ vector space span the representation $(\mathbf{2}, \mathbf{2}, \mathbf{2})$, while the directions $\{5, 6, 7, 12, 13, 14\}$ of the same space span the representations $(\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2})$. The adjoint representation of $\mathfrak{sp}(6, \mathbb{R})$ decomposes instead in the following way:

$$\begin{aligned} \text{adj} [\mathfrak{sp}(6, \mathbb{R})] &\xrightarrow{\mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2)} (\mathbf{3}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{3}) \\ &\oplus (\mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}) \end{aligned} \quad (7.43)$$

as it is evident by a quick inspection of the roots (7.17). In terms of the Cartan-Weyl basis the three $\mathfrak{sl}(2, \mathbb{R})$ subalgebra contains the three Cartan generators \mathcal{H}_i and the step operators $\mathcal{E}^{\pm\alpha_3}, \mathcal{E}^{\pm\alpha_7}, \mathcal{E}^{\pm\alpha_9}$. The remaining 12 step operators span the representation $(\mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})$, namely:

$$(\mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2}) = \text{span} [\mathcal{E}^{\pm\alpha_1}, \mathcal{E}^{\pm\alpha_2}, \mathcal{E}^{\pm\alpha_4}, \mathcal{E}^{\pm\alpha_5}, \mathcal{E}^{\pm\alpha_6}, \mathcal{E}^{\pm\alpha_8}] \quad (7.44)$$

The explicit form of an $\mathfrak{sp}(6, \mathbb{R})$ Lie algebra element reduced to the $\mathfrak{sl}(2)^3$ subalgebra is the following one:

$$\begin{pmatrix} h_1 & 0 & 0 & b_1 & 0 & 0 \\ 0 & h_2 & 0 & 0 & b_2 & 0 \\ 0 & 0 & h_3 & 0 & 0 & b_3 \\ c_1 & 0 & 0 & -h_1 & 0 & 0 \\ 0 & c_2 & 0 & 0 & -h_2 & 0 \\ 0 & 0 & c_3 & 0 & 0 & -h_3 \end{pmatrix} \in \mathfrak{sl}(2) \otimes \mathfrak{sl}(2) \otimes \mathfrak{sl}(2) \subset \mathfrak{sp}(6, \mathbb{R}) \quad (7.45)$$

7.7 REDUCTION OF THE CHARGE VECTOR TO THE $(2, 2, 2)$

In order to study the orbits of the charge vectors in the $\mathbf{14}'$ our first step consists of reducing it to normal form, namely to the $(2, 2, 2)$ representation. We claim that for generic charge vectors this is always possible by means of $\mathrm{Sp}(6, \mathbb{R})$ rotations generated by elements of the $(\mathbf{2}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{2})$ subspace. To show this let us consider the six dimensional compact Lie algebra element:

$$\begin{aligned} \mathbb{K}_\psi &= \psi_1 (\mathcal{E}^{\alpha_1} - \mathcal{E}^{-\alpha_1}) + \psi_2 (\mathcal{E}^{\alpha_2} - \mathcal{E}^{-\alpha_2}) + \psi_3 (\mathcal{E}^{\alpha_4} - \mathcal{E}^{-\alpha_4}) \\ &\quad \psi_4 (\mathcal{E}^{\alpha_5} - \mathcal{E}^{-\alpha_5}) + \psi_5 (\mathcal{E}^{\alpha_6} - \mathcal{E}^{-\alpha_6}) + \psi_6 (\mathcal{E}^{\alpha_8} - \mathcal{E}^{-\alpha_8}) \end{aligned} \quad (7.46)$$

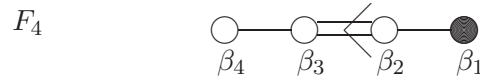
and a generic charge vector that has components only in the $(\mathbf{2}, \mathbf{2}, \mathbf{2})$ subspace.

$$\mathcal{Q}_{2,2,2} = \{\Theta_1, \Theta_2, \Theta_3, \Theta_4, 0, 0, 0, \Theta_5, \Theta_6, \Theta_7, \Theta_8, 0, 0, 0\} \quad (7.47)$$

If we apply the $\mathbf{14}'$ representation of \mathbb{K}_ψ to the charge vector \mathcal{Q}_N we obtain:

$$\mathcal{D}_{14}(\mathbb{K}_\psi) \mathcal{Q}_{2,2,2} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -\sqrt{2}\Theta_2\psi_2 + \sqrt{2}\Theta_5\psi_2 - \sqrt{2}\Theta_4\psi_4 - \sqrt{2}\Theta_7\psi_4 \\ -\sqrt{2}\Theta_3\psi_3 - \sqrt{2}\Theta_5\psi_3 + \sqrt{2}\Theta_4\psi_5 - \sqrt{2}\Theta_6\psi_5 \\ \sqrt{2}\Theta_2\psi_1 + \sqrt{2}\Theta_3\psi_1 - \sqrt{2}\Theta_1\psi_6 - \sqrt{2}\Theta_4\psi_6 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\sqrt{2}\Theta_1\psi_2 - \sqrt{2}\Theta_6\psi_2 + \sqrt{2}\Theta_3\psi_4 - \sqrt{2}\Theta_8\psi_4 \\ \sqrt{2}\Theta_1\psi_3 - \sqrt{2}\Theta_7\psi_3 + \sqrt{2}\Theta_2\psi_5 + \sqrt{2}\Theta_8\psi_5 \\ \sqrt{2}\Theta_6\psi_1 + \sqrt{2}\Theta_7\psi_1 - \sqrt{2}\Theta_5\psi_6 - \sqrt{2}\Theta_8\psi_6 \end{pmatrix} \quad (7.48)$$

which clearly shows that the six parameters ψ_1, \dots, ψ_6 are sufficient to generate arbitrary components $\{5, 6, 7, 12, 13, 14\}$ of the charge vector starting from vanishing ones. Reverting the path this means that by means of the same



$$\begin{aligned}\psi &= \beta_{24} = 2\beta_1 + 3\beta_2 + 4\beta_3 + 2\beta_4 \\ (\psi, \beta_1) &= 2 \quad ; \quad (\psi, \beta_i) = 0 \quad i \neq 1\end{aligned}$$

Figure 2: The Dynkin diagram of $F_{4(4)}$. The only root which is not orthogonal to the highest root is $\beta_V = \beta_1$. The root $\beta_V = \beta_1$ is the highest weight of the **W**-representation of $\mathfrak{sp}(6, \mathbb{R})$

rotations, apart from singular orbits that deserve a separate study we can always fix the gauge where the six components $\{5, 6, 7, 12, 13, 14\}$ vanish.

7.7.1 FURTHER REDUCTION TO NORMAL FORM OF THE CHARGE VECTOR

Once the charge vector is reduced to $(2, 2, 2)$ representation, we can further act on it with the $\text{SL}(2, \mathbb{R})^3$ group in order to further reduce its components. By using the three parameters of the abelian translation group \mathbb{R}^3 contained in $\text{SL}(2, \mathbb{R})^3$ we can put to zero three of the eight charges and a possible normal form of the charge vector is the following one:

$$\mathcal{Q}_N = \{0, P_1, P_2, P_3, 0, 0, 0, P_4, 0, 0, P_5, 0, 0, 0\} \quad (7.49)$$

The corresponding quartic invariant is:

$$\mathfrak{I}_4(\mathcal{Q}_N) = P_3^2 P_5^2 - 4P_1 P_2 P_4 P_5 \quad (7.50)$$

8 THE $\frac{F_{(4,4)}}{\text{SU}(2) \times \text{USp}(6)}$ QUATERNIONIC KÄHLER MANIFOLD

Let us now come to the c -map image of the Special Kähler manifold (7.1), namely to the quaternionic Kähler manifold (7.2). The $F_{(4,4)}$ Lie algebra has rank four and its structure is codified in the Dynkin diagram presented in fig.2. The complete set of positive roots contains 24 elements that are listed below:

$$\begin{array}{lll}
\beta_1 & = & \beta_1 & = & \{-1, -1, -1, 1\} \\
\beta_2 & = & \beta_2 & = & \{0, 0, 2, 0\} \\
\beta_3 & = & \beta_3 & = & \{0, 1, -1, 0\} \\
\beta_4 & = & \beta_4 & = & \{1, -1, 0, 0\} \\
\beta_5 & = & \beta_1 + \beta_2 & = & \{-1, -1, 1, 1\} \\
\beta_6 & = & \beta_2 + \beta_3 & = & \{0, 1, 1, 0\} \\
\beta_7 & = & \beta_3 + \beta_4 & = & \{1, 0, -1, 0\} \\
\beta_8 & = & \beta_1 + \beta_2 + \beta_3 & = & \{-1, 0, 0, 1\} \\
\beta_9 & = & \beta_2 + 2\beta_3 & = & \{0, 2, 0, 0\} \\
\beta_{10} & = & \beta_2 + \beta_3 + \beta_4 & = & \{1, 0, 1, 0\} \\
\beta_{11} & = & \beta_1 + \beta_2 + 2\beta_3 & = & \{-1, 1, -1, 1\} \\
\beta_{12} & = & \beta_1 + \beta_2 + \beta_3 + \beta_4 & = & \{0, -1, 0, 1\} \\
\beta_{13} & = & \beta_2 + 2\beta_3 + \beta_4 & = & \{1, 1, 0, 0\} \\
\beta_{14} & = & \beta_1 + 2\beta_2 + 2\beta_3 & = & \{-1, 1, 1, 1\} \\
\beta_{15} & = & \beta_1 + \beta_2 + 2\beta_3 + \beta_4 & = & \{0, 0, -1, 1\} \\
\beta_{16} & = & \beta_2 + 2\beta_3 + 2\beta_4 & = & \{2, 0, 0, 0\} \\
\beta_{17} & = & \beta_1 + 2\beta_2 + 2\beta_3 + \beta_4 & = & \{0, 0, 1, 1\} \\
\beta_{18} & = & \beta_1 + \beta_2 + 2\beta_3 + 2\beta_4 & = & \{1, -1, -1, 1\} \\
\beta_{19} & = & \beta_1 + 2\beta_2 + 3\beta_3 + \beta_4 & = & \{0, 1, 0, 1\} \\
\beta_{20} & = & \beta_1 + 2\beta_2 + 2\beta_3 + 2\beta_4 & = & \{1, -1, 1, 1\} \\
\beta_{21} & = & \beta_1 + 2\beta_2 + 3\beta_3 + 2\beta_4 & = & \{1, 0, 0, 1\} \\
\beta_{22} & = & \beta_1 + 2\beta_2 + 4\beta_3 + 2\beta_4 & = & \{1, 1, -1, 1\} \\
\beta_{23} & = & \beta_1 + 3\beta_2 + 4\beta_3 + 2\beta_4 & = & \{1, 1, 1, 1\} \\
\beta_{24} & = & 2\beta_1 + 3\beta_2 + 4\beta_3 + 2\beta_4 & = & \{0, 0, 0, 2\}
\end{array} \tag{8.1}$$

In eq.(8.1) the first column is the name of the root, the second column gives its decomposition in terms of simple roots, while the last column provides the component of the root vector in \mathbb{R}^4 .

The standard Cartan-Weyl form of the Lie algebra is as follows:

$$[\mathcal{H}_i, E^{\pm\beta}] = \pm \beta^i E^{\pm\beta_I} \tag{8.2}$$

$$[E^\beta, E^{-\beta}] = \beta \cdot \mathcal{H} \tag{8.3}$$

$$[E^\beta, E^\gamma] = \begin{cases} N_{\beta\gamma} E^{\beta+\gamma} & \text{if } \beta + \gamma \text{ is a root} \\ 0 & \text{if } \beta + \gamma \text{ is not a root} \end{cases} \tag{8.4}$$

where $N_{\beta\gamma}$ are numbers that can be predicted from Lie algebra theory. They are irrelevant, since they follow from commutators, when one has an explicit matrix realization of all the Cartan generators \mathcal{H}_i and of the step operators $E^{\pm\beta}$. The fundamental representation of $F_{(4,4)}$ is 26-dimensional and real. We have constructed in

a MATHEMATICA code all the 26×26 matrix representations of the 52 generators of the Cartan Weyl basis. Obviously we can not present them here because they are too big. However all the rest of the construction can be easily presented in terms of these Weyl generators sand this is what we presently do.

8.1 THE MAXIMAL COMPACT SUBALGEBRA $\mathbb{H} = \mathfrak{su}(2) \oplus \mathfrak{usp}(6)$

The maximal compact subalgebra \mathbb{H} of a maximally split simple Lie algebra such as $F_{(4,4)}$, is just the real span of all the independent compact generators $E^{\beta_i} - E^{-\beta_i}$. In our case we have 24 positive roots and we can write:

$$\mathbb{H} = \text{span}_{\mathbb{R}} \{H_1, H_2, \dots, H_{24}\} \quad (8.5)$$

where we have defined:

$$H_i = E^{\beta_i} - E^{-\beta_i} \quad (8.6)$$

the positive roots being numbered as in eq.(8.1). We know from theory that this maximal compact subalgebra has the structure:

$$\mathbb{H} = \mathfrak{su}(2) \oplus \mathfrak{usp}(6) \quad (8.7)$$

It is important to derive an explicit basis of generators satisfying the standard commutation relations of the two simple factors in eq.(8.7) for holonomy calculations of the coset manifold. Particularly important are the three generators J^x of the $\mathfrak{su}(2)$ subalgebra since they will act as quaternionic complex structures in the calculation of the tri-holomorphic moment map. By means of standard techniques of diagonalization of the adjoint action of generators we have retrieved the required basis rearrangement.

8.1.1 THE $\mathfrak{su}(2)$ LIE ALGEBRA

The three generators J^x have tho following explicit form:

$$\begin{aligned} J^1 &= \frac{H_1 - H_{14} + H_{20} - H_{22}}{4\sqrt{2}} \\ J^2 &= \frac{H_5 + H_{11} - H_{18} + H_{23}}{4\sqrt{2}} \\ J^3 &= -\frac{H_2 - H_9 + H_{16} + H_{24}}{4\sqrt{2}} \end{aligned} \quad (8.8)$$

and close the standard commutation relations:

$$[J^x, J^y] = \epsilon^{xyz} J^z \quad (8.9)$$

8.1.2 THE $\mathfrak{usp}(6)$ LIE ALGEBRA

The 21 generators of the $\mathfrak{usp}(6)$ Lie algebra are given by the following combinations. First we have three mutually commuting generators (the compact Cartan generators):

$$[\mathcal{L}^i, \mathcal{L}^j] = 0 \quad (8.10)$$

that are given by the following combinations:

$$\begin{aligned} \mathcal{L}^1 &= -\frac{H_2}{2} - \frac{H_9}{2} + \frac{H_{16}}{2} - \frac{H_{24}}{2} \\ \mathcal{L}^2 &= -\frac{H_2}{2} + \frac{H_9}{2} + \frac{H_{16}}{2} + \frac{H_{24}}{2} \\ \mathcal{L}^3 &= \frac{H_2}{2} + \frac{H_9}{2} + \frac{H_{16}}{2} - \frac{H_{24}}{2} \end{aligned} \quad (8.11)$$

Secondly we have 9 pairs of generators $\{X_i, Y_i\}$ which are in correspondence with the 9 positive roots of the $\mathfrak{sp}(6, C)$ Lie algebra (see eq.(7.17). Explicitly we have:

$$\begin{aligned}
X_1 &= H_{10} & ; & Y_1 = H_7 \\
X_2 &= H_4 & ; & Y_2 = -H_{13} \\
X_3 &= H_6 & ; & Y_3 = -H_3 \\
X_4 &= -H_1 + H_{14} + H_{20} - H_{22} & ; & Y_4 = -H_5 - H_{11} - H_{18} + H_{23} \\
X_5 &= H_{21} & ; & Y_5 = -H_8 \\
X_6 &= H_1 + H_{14} + H_{20} + H_{22} & ; & Y_6 = H_5 - H_{11} - H_{18} - H_{23} \\
X_7 &= -H_1 - H_{14} + H_{20} + H_{22} & ; & Y_7 = H_5 - H_{11} + H_{18} + H_{23} \\
X_8 &= H_{17} & ; & Y_8 = H_{15} \\
X_9 &= H_{12} & ; & Y_9 = H_{19}
\end{aligned} \tag{8.12}$$

The commutation relations with the compact Cartan generators are as follows:

$$[\mathcal{L}^i, X_I] = \alpha_I^i Y_I \quad ; \quad [\mathcal{L}^i, Y_I] = -\alpha_I^i X_I \tag{8.13}$$

where α_I are the roots of eq.(7.17). The remaining commutation relations mix the Y and the X among themselves and reproduce the Cartan generators.

8.2 THE SUBALGEBRA $\mathfrak{sl}(2, \mathbb{R})_E \oplus \mathfrak{sp}(6, \mathbb{R})$ AND THE \mathbf{W} -GENERATORS

Of great relevance in all applications of the (pseudo)-quaternionic geometry either in the construction of Black-Hole solutions or in the quest of inflaton potentials by means of the gauging of hypermultiplet isometries is the identification of the subalgebra:

$$\mathfrak{sl}(2, \mathbb{R})_E \oplus \mathfrak{sp}(6, \mathbb{R}) \subset \mathfrak{f}_{(4,4)} \tag{8.14}$$

and the recasting of $\mathfrak{f}_{(4,4)}$ in the general form 4.53 by means of the identification of the \mathbf{W} -generators.

To this effect a very powerful tool is provided by the comparison of the $\mathfrak{f}_{(4,4)}$ root system displayed in eq.(8.1) with the $\mathfrak{sp}(6, \mathbb{R})$ root system displayed in eq.(7.17). The step operators associated with the highest (lowest) root $\pm\beta_{24}$ are the only ones that have a grading ± 2 with respect to the fourth Cartan generator \mathcal{H}_4 . These three operators close among themselves the Lie algebra $\mathfrak{sl}(2, \mathbb{R})_E$. There are 9 roots that have grading zero with respect to \mathcal{H}_4 . Projected onto the plane $\mathcal{H}_4 = 0$ these 9 roots form, together with their negatives, a $\mathfrak{sp}(6, \mathbb{R})$ root system. Correspondingly the $\mathfrak{sp}(6, \mathbb{R})$ subalgebra is generated by the step operators associated with these 9 roots (and with their negatives) plus the first 3 Cartan generators. Finally there are 14 positive roots β that have have grading 1 with respect to \mathcal{H}_4 . The step operators associated with these 14 roots form the \mathbf{W} -generators with index 1 of $\text{SL}(2, \mathbb{R})_E$. Their partners with index 2 are provided by the corresponding negative root step operators.

It is quite important to arrange the generators \mathbf{W} in such a way that under any element $\mathfrak{g} \in \mathfrak{sp}(6, \mathbb{R}) \subset \mathfrak{f}_{(4,4)}$ they transform exactly with the $\mathcal{D}_{14}(\mathfrak{g})$ matrices defined in eq.s(A.11), (A.12) and (A.13).

The precise definition of all the generators that satisfy the specified requirements is given below.

8.2.1 THE EHLERS SUBALGEBRA $\mathfrak{sl}(2, \mathbb{R})_E$.

The standard commutation relations:

$$[L_0^E, L_\pm^E] = \pm L_\pm^E \quad (8.15)$$

$$[L_+^E, L_-^E] = 2 L_0^E \quad (8.16)$$

are satisfied by the following generators:

$$\begin{aligned} L_0^E &= \frac{1}{2} \mathcal{H}_4 \\ L_+^E &= \frac{1}{\sqrt{2}} \mathcal{E}^{\beta_{24}} \\ L_-^E &= \frac{1}{\sqrt{2}} \mathcal{E}^{-\beta_{24}} \end{aligned} \quad (8.17)$$

8.2.2 THE SUBALGEBRA $\mathfrak{sp}(6, \mathbb{R})$.

The Cartan generators are the following ones:

$$\begin{aligned} \mathcal{H}_1 &= \mathcal{H}_1 \\ \mathcal{H}_2 &= \mathcal{H}_2 \\ \mathcal{H}_3 &= \mathcal{H}_3 \end{aligned} \quad (8.18)$$

while the step operators are identified as follows

$$\begin{aligned} \mathcal{E}^{\pm\alpha_1} &= \mathcal{E}^{\pm\beta_4} \\ \mathcal{E}^{\pm\alpha_2} &= \mathcal{E}^{\pm\beta_3} \\ \mathcal{E}^{\pm\alpha_3} &= \mathcal{E}^{\pm\beta_2} \\ \mathcal{E}^{\pm\alpha_4} &= \mathcal{E}^{\pm\beta_7} \\ \mathcal{E}^{\pm\alpha_5} &= -\mathcal{E}^{\pm\beta_6} \\ \mathcal{E}^{\pm\alpha_6} &= \mathcal{E}^{\pm\beta_{10}} \\ \mathcal{E}^{\pm\alpha_7} &= -\mathcal{E}^{\pm\beta_9} \\ \mathcal{E}^{\pm\alpha_8} &= \mathcal{E}^{\pm\beta_{13}} \\ \mathcal{E}^{\pm\alpha_9} &= \mathcal{E}^{\pm\beta_{16}} \end{aligned} \quad (8.19)$$

We would like to attract the attention of the reader on the two minus signs introduced in the identifications (8.19). Together with the other minus signs that appear below in the identification of the W -generators these signs are essential in order for the transformations of the W .s to be identical with those given by the previously defined $\mathcal{D}_{14}(\mathfrak{g})$ matrices.

8.2.3 THE \mathbf{W} -GENERATORS

Casting the $\mathfrak{f}_{(4,4)}$ Lie algebra in the general form (4.53) is completed by the identification of the \mathbf{W} -generators. We find:

$$\begin{aligned}
\mathbf{W}^{1,1} &= \mathcal{E}^{\beta_5} \\
\mathbf{W}^{1,2} &= \mathcal{E}^{\beta_{20}} \\
\mathbf{W}^{1,3} &= \mathcal{E}^{\beta_{14}} \\
\mathbf{W}^{1,4} &= -\mathcal{E}^{\beta_{23}} \\
\mathbf{W}^{1,5} &= \mathcal{E}^{\beta_{21}} \\
\mathbf{W}^{1,6} &= \mathcal{E}^{\beta_{19}} \\
\mathbf{W}^{1,7} &= -\mathcal{E}^{\beta_{17}} \\
\mathbf{W}^{1,8} &= -\mathcal{E}^{\beta_{22}} \\
\mathbf{W}^{1,9} &= -\mathcal{E}^{\beta_{11}} \\
\mathbf{W}^{1,10} &= -\mathcal{E}^{\beta_{18}} \\
\mathbf{W}^{1,11} &= -\mathcal{E}^{\beta_1} \\
\mathbf{W}^{1,12} &= -\mathcal{E}^{\beta_8} \\
\mathbf{W}^{1,13} &= -\mathcal{E}^{\beta_{12}} \\
\mathbf{W}^{1,14} &= -\mathcal{E}^{\beta_{15}}
\end{aligned} \tag{8.20}$$

and for all $\mathfrak{g} \in \mathfrak{sp}(6, \mathbb{R}) \subset \mathfrak{f}_{(4,4)}$ we have:

$$[\mathfrak{g}, \mathbf{W}^{1,\alpha}] = \mathcal{D}_{14}(\mathfrak{g})^\alpha{}_\gamma \mathbf{W}^{1,\gamma} \tag{8.21}$$

The generators $\mathbf{W}^{2,\alpha}$ are then easily obtained from by means of a rotation with the unique compact generator of the Ehlers subalgebra introduced in eq.(4.56):

$$[\mathfrak{E}, \mathbf{W}^{1,\alpha}] = \mathbf{W}^{2,\alpha} \tag{8.22}$$

8.3 THE SOLVABLE COSET REPRESENTATIVE

The precise constructions of the previous sections enable us to introduce the solvable coset representative $\mathbb{L}_{Solv}(a, U, h, p, Z)$ of the manifold (7.2) such that the Maurer Cartan form:

$$\Xi \equiv \mathbb{L}_{Solv}^{-1} d\mathbb{L}_{Solv} \tag{8.23}$$

decomposed along the generators of the Borel Lie algebra:

$$\begin{aligned}
\Xi &= E_{\mathcal{QM}}^I T_I \\
T_I &= \left\{ \underbrace{L_0^E, L_+^E}_{2 \hookrightarrow Solv[\mathfrak{sl}(2)]}, \underbrace{\mathcal{H}^i, \mathcal{E}^{\alpha_i}}_{12 \hookrightarrow Solv[\mathfrak{sp}(6)]}, \underbrace{\mathbf{W}^{1\alpha}}_{14 \hookrightarrow \mathbb{H}eis} \right\}
\end{aligned} \tag{8.24}$$

provides the vielbein $E_{\mathcal{QM}}^I$ mentioned in eq.(4.40) and by squaring the metric (4.41).

In full analogy with eq.s (6.36) and (7.32) we write:

$$\begin{aligned} \mathbb{L}_{Solv} = & \exp[a L_+^E] \cdot \exp\left[\sum_{j=1}^7 \mathbf{Z}_{2j-1} \mathbf{W}^{1,2j-1}\right] \cdot \exp\left[\sum_{j=1}^7 \mathbf{Z}_{2j} \mathbf{W}^{1,2j}\right] \times \\ & \times \prod_{i=1}^9 \exp[p_{10-i} \mathcal{E}^{\alpha_{10-i}}] \cdot \prod_{j=3}^3 \exp[h_j \mathcal{H}^j] \cdot \exp[U L_0^E] \end{aligned} \quad (8.25)$$

The explicit expression of \mathbb{L}_{Solv} in the fundamental 26-dimensional representation is obviously very large but it can be dealt with by means of an appropriate MATHEMATICA code.

We are finally in the position of calculating the tri-holomorphic moment map of any element $\mathfrak{t} \in \mathfrak{f}_{(4,4)}$ of the isometry Lie algebra of \mathcal{QM} through the formula:

$$\mathcal{P}_{\mathfrak{t}}^x = \text{Tr}_{\mathbf{26}} (J^x \mathbb{L}_{Solv}^{-1} \mathfrak{t} \mathbb{L}_{Solv}) \quad (8.26)$$

8.4 THE EXAMPLE OF THE INCLUSION OF MULTI STAROBINSKY MODELS

In section 7.6 we studied the truncation of the $\mathfrak{sp}(6, \mathbb{R})$ model to the STU model. There we showed that setting to zero the three complex coordinates z_2, z_3, z_5 , the remaining ones z_1, z_4, z_6 span the STU model, namely they parameterize three copies of the Lobachevsky-Poincaré hyperbolic plane. Inspecting eq. (7.35) we also see that the three coordinates z_1, z_4, z_6 are the only ones that survive when all the axions p_i are set to zero. We also recall from sect. 7.6 that the three parabolic generators of the three $\text{SL}(2, \mathbb{R})$ groups spanning the STU model are $\mathcal{E}^{\alpha_3}, \mathcal{E}^{\alpha_7}, \mathcal{E}^{\alpha_9}$ whose identification with $\mathfrak{f}_{(4,4)}$ generators is provided by eq.(8.19). Correspondingly we introduce the following generator:

$$\mathfrak{t}_{STU} = \beta_3 \mathcal{E}^{\alpha_3} + \beta_2 \mathcal{E}^{\alpha_7} + \beta_1 \mathcal{E}^{\alpha_9} - \kappa \mathfrak{S} \quad (8.27)$$

and we calculate its tri-holomorphic moment map, by means of eq.(8.26). Defining the potential:

$$V_{STU} = \sum_{x=1}^3 (\mathcal{P}_{\mathfrak{t}_{STU}}^x)^2 \quad (8.28)$$

We can verify that:

$$\begin{aligned} \left. \frac{\partial}{\partial \mathbf{Z}^\alpha} V_{STU} \right|_{\mathbf{Z}=U=a=0} &= 0 \\ \left. \frac{\partial}{\partial U} V_{STU} \right|_{\mathbf{Z}=U=a=0} &= 0 \\ \left. \frac{\partial}{\partial a} V_{STU} \right|_{\mathbf{Z}=U=a=0} &= 0 \end{aligned} \quad (8.29)$$

Hence we can consistently truncate U, a and the Heisenberg fields \mathbf{Z} . We find:

$$V_{STU}|_{\mathbf{Z}=U=a=0} = \frac{9}{4} \left(2\kappa - \sqrt{2} \sum_{i=1}^3 \beta_i e^{-2h_i} \right)^2 \quad (8.30)$$

The above potential can be named a multi-Starobinsky model with three independent dilatons.

First of all let us note that in the above model the absolute value of β_i is irrelevant since we can always reabsorb it by a constant shift $h_i \rightarrow h_i - \log |\beta_i|$. The only relevant thing are the signs of β_i including in this notion also zero, namely β_i can be ± 1 or 0. Secondly we observe that when all the non vanishing β_i have the same sign we can make a consistent one field truncation to

$$h_i = h \quad ; \quad \text{for all } i \text{ such that } \beta_i \neq 0 \quad (8.31)$$

After this truncation the potential (8.30) becomes the following:

$$V_{eff} = \frac{9}{4} \left(2\kappa - \sqrt{2} q e^{-2h} \right)^2 \quad (8.32)$$

where q is the number of equal sign non zero β_i , which obviously can take only three values $q = 1, 2, 3$. In order to compare this result with the definition of α -attractors introduced in [5], we just have to compare the potential (8.32) with the normalization of the scalar kinetic terms in the lagrangian:

$$\mathcal{L} = \dots + \frac{1}{4} (\partial U)^2 + (\partial h_1)^2 + (\partial h_2)^2 + (\partial h_3)^2 + \dots \quad (8.33)$$

which follows from eq.s(4.4, A.14). Renaming $h = \frac{1}{\sqrt{2}q} \phi$, so that the new field ϕ has canonical kinetic term $\frac{1}{2} (\partial \phi)^2$, we obtain a potential:

$$V_{eff} = \text{const} \times \left(2\kappa - \sqrt{2} q \exp \left[-\sqrt{\frac{2}{q}} \phi \right] \right)^2 \quad (8.34)$$

which, in the notation of [5], corresponds to $\alpha = \frac{q}{3}$, namely to:

$$\alpha = 1, \frac{2}{3}, \frac{1}{3} \quad (8.35)$$

The above result has been obtained by gauging only one generator, namely (8.27). Correspondingly we have generated Starobinsky-like models with only one massive vector that is the gauge vector associated with the gauged generator. There is another way of obtaining the same potential but with q -massive vectors (one for each constituent Starobinsky model with $q = \frac{1}{3}$). This is very simply understood remarking that the $\mathfrak{f}_{(4,4)}$ algebra contains an $\mathfrak{sl}(2, \mathbb{R})^4$ subalgebra singled out as follows:

$$\mathfrak{f}_{(4,4)} \supset \mathfrak{sl}(2, \mathbb{R})_E \oplus \underbrace{\mathfrak{sl}(2, \mathbb{R})_S \oplus \mathfrak{sl}(2, \mathbb{R})_T \oplus \mathfrak{sl}(2, \mathbb{R})_U}_{\subset \mathfrak{sp}(6, \mathbb{R})} \quad (8.36)$$

where $\mathfrak{sl}(2, \mathbb{R})_S \oplus \mathfrak{sl}(2, \mathbb{R})_T \oplus \mathfrak{sl}(2, \mathbb{R})_U$ describes the STU model embedded in the Kähler manifold (7.1). These four $\mathfrak{sl}(2, \mathbb{R})$ algebras are completely symmetric among themselves and the gauging of their generators produce identical results. So we can introduce the abelian gauge algebra spanned by the following three commuting generators:

$$\begin{aligned} \mathfrak{t}_S &= \beta_3 \mathcal{E}^{\alpha_3} - \kappa_3 \mathfrak{S} \\ \mathfrak{t}_T &= \beta_2 \mathcal{E}^{\alpha_7} - \kappa_2 \mathfrak{S} \\ \mathfrak{t}_U &= \beta_1 \mathcal{E}^{\alpha_9} - \kappa_1 \mathfrak{S} \end{aligned} \quad (8.37)$$

Gauging with three separate vectors each of the above generators we obtain a new potential:

$$\widehat{V}_{STU} = \sum_{x=1}^3 (\mathcal{P}_{t_S}^x)^2 + \sum_{x=1}^3 (\mathcal{P}_{t_T}^x)^2 + \sum_{x=1}^3 (\mathcal{P}_{t_U}^x)^2 \quad (8.38)$$

that has the same property as the potential (8.28), namely it allows us to truncate consistently to zero all the axions p_i , all the Heisenberg fields \mathbf{Z}^α and the Taub NUT field a . The reduced potential after such a truncation has the form:

$$\widehat{V}_{red} = \frac{9}{4} \sum_{i=1}^3 \left(2\kappa_i - \sqrt{2}e^{-2h_i}\beta_i \right)^2 \quad (8.39)$$

As we already remarked before, the absolute value of the β_i parameters is irrelevant: what matters is only the relative signs of the β_i with respect to the sign of their corresponding κ_i . If for all non vanishing β_i we have $\frac{\beta_i}{\kappa_i} = 1$, then we can consistently perform the same truncation (8.31) as before and we reobtain the potentials (8.34) with the same spectrum of α -values (8.35). The difference with the previous case is, as we emphasized at the beginning of this discussion, that now the number of massive fields is q , namely as many as the elementary non trivial constituent Starobinsky-like models.

8.5 NILPOTENT GAUGINGS AND TRUNCATIONS

Let us now put the above obtained results in the general framework discussed in sect.5.2. The issue is the classification of orbits of nilpotent operators and the question whether for each of these orbits we can find a consistent one-field reduction that produces a Starobinsky-like model with an appropriate value of α .

To answer this question we have followed the algorithm described in the second paper of [4]. According to a general mathematical set, up to conjugation, every nilpotent orbit is associated with a standard triple $\{x, y, h\}$ satisfying the standard commutation relations of the $\mathfrak{sl}(2)$ Lie algebra, namely:

$$[h, x] = x \quad ; \quad [h, y] = -y \quad ; \quad [x, y] = 2h \quad (8.40)$$

Interesting for us is the classification of nilpotent orbits in the Kähler subalgebra $\mathfrak{sp}(6, \mathbb{R})$ and, according to the above mathematical theory, this is just the classification of embeddings of an $\mathfrak{sl}(2)$ Lie algebra in the ambient one, modulo conjugation by the full group $\mathrm{Sp}(6, \mathbb{R})$. The second relevant point emphasized in [4] is that embeddings of subalgebras $\mathfrak{h} \subset \mathfrak{g}$ are characterized by the branching law of any representation of \mathfrak{g} into irreducible representations of \mathfrak{h} . Clearly two embeddings might be conjugate only if their branching laws are identical. Embeddings with different branching laws necessarily belong to different orbits. In the case of the $\mathfrak{sl}(2) \sim \mathfrak{so}(1, 2)$ Lie algebra, irreducible representations are uniquely identified by their spin j , so that the branching law is expressed by listing the angular momenta $\{j_1, j_2, \dots, j_n\}$ of the irreducible blocks into which any representation of the original algebra, for instance the fundamental, decomposes with respect to the embedded subalgebra. The dimensions of each irreducible module is $2j + 1$ so that an a priori constraint on the labels $\{j_1, j_2, \dots, j_n\}$ characterizing an irreducible orbit of $\mathfrak{sp}(6, \mathbb{R})$ is the summation rule:

$$\sum_{i=1}^n (2j_i + 1) = 6 = \text{dimension of the fundamental representation} \quad (8.41)$$

Therefore we have considered all possible partitions of the number 6 into integers and for each partition we have constructed a candidate h element in the Cartan subalgebra of $\mathfrak{sp}(6, \mathbb{R})$ containing as eigenvalues all

the J_3 values of the corresponding $\{j_1, j_2, \dots, j_n\}$ representation. To clarify what we mean by this it suffices to consider the example of the first partition $6 = 6$. In this case the 6 dimensional representation of $\mathfrak{sl}(2)$ is the $j = \frac{5}{2}$ and the 6 eigenvalues are $\pm\frac{5}{2}, \pm\frac{3}{2}, \pm\frac{1}{2}$. Having so fixed the so named central element h of the candidate standard triplet we have tried to construct the corresponding x and y . Imposing the standard commutation relations (8.40) one obtains quadratic equations on the coefficients of the linear combinations expressing the candidate x and y that may have or may not have solutions. If the solutions exist, then the corresponding standard triple is found, the orbit exists and we have constructed one representative x .

Next, given the existing orbits and the corresponding standard triples, for each of them we have constructed a Lobachevsky complex plane immersed in the Special Kähler manifold \mathcal{M}_{Sp6} defined by eq.(7.1). The construction is very simple. One calculates the group element $\mathbf{g}(\lambda, \psi) \in \mathfrak{sp}(6, \mathbb{R})$ defined below:

$$\mathbf{g}(\lambda, \psi) = \exp[\psi x] \cdot \exp[\lambda h] = \left(\begin{array}{c|c} \mathbf{A}(\lambda, \psi) & \mathbf{B}(\lambda, \psi) \\ \hline \mathbf{C}(\lambda, \psi) & \mathbf{D}(\lambda, \psi) \end{array} \right) \quad (8.42)$$

and using equation (7.8), we write:

$$\begin{aligned} Z(\lambda, \psi) &= (\mathbf{A}(\lambda, \psi) - i\mathbf{B}(\lambda, \psi)) \cdot (\mathbf{C}(\lambda, \psi) - i\mathbf{D}(\lambda, \psi))^{-1} \\ &\equiv \begin{pmatrix} z_1(\lambda, \psi) & z_2(\lambda, \psi) & z_3(\lambda, \psi) \\ z_2(\lambda, \psi) & z_4(\lambda, \psi) & z_5(\lambda, \psi) \\ z_3(\lambda, \psi) & z_5(\lambda, \psi) & z_6(\lambda, \psi) \end{pmatrix} \end{aligned} \quad (8.43)$$

which defines the explicit embedding:

$$\phi : \frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{SO}(2)} \rightarrow \frac{\mathrm{Sp}(6, \mathbb{R})}{\mathrm{SU}(3) \times \mathrm{U}(1)} \equiv \mathcal{M}_{Sp6} \quad (8.44)$$

of the Lobachevsky plane in \mathcal{M}_{Sp6} . Indeed from (8.43) we read off the parameterization of the complex coordinates z_i ($i = 1, \dots, 6$) as functions of $\lambda = \log \mathrm{Im} w$ and $\psi = \mathrm{Re} w$, the complex variable w being the local variable over the embedded Poincaré-Lobachevsky plane.

The question is whether the field equations of the scalar fields:

$$\partial_i \partial_{j^*} \mathcal{K} \partial^\mu \partial_\mu \bar{z}^{j^*} + \partial_i \partial_{j^*} \partial_{k^*} \mathcal{K} \partial_\mu z^{j^*} \partial_\mu z^{k^*} - \frac{1}{4} \partial_i V_{gauging}(z, \bar{z}) = 0 \quad (8.45)$$

admit first a consistent reduction to the complex scalar field w and then a consistent truncation to a vanishing axion $\psi = 0$. Consistency of the truncation can be verified or disproved in the following simple way. The pull-back on the immersed surface $\phi^* \left(\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{SO}(2)} \right) \subset \mathcal{M}_{Sp6}$ of the twelve field equations (8.45) (six complex equations) should be consistent among themselves and be identical with the two field equations obtained from the variation of the pull-back $\phi^*(\mathcal{L})$ on the immersed surface of the Lagrangian \mathcal{L} from which eq.s (8.45) derive, namely:

$$\mathcal{L} = 4 \partial_i \partial_{j^*} \mathcal{K} \partial_\mu z^i \partial^\mu \bar{z}^{j^*} - V_{gauging}(z, \bar{z}) \quad (8.46)$$

In other words, defining $w = i e^\lambda + \psi$, the truncation is consistent if the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{L}(z, \bar{z}) & \xrightarrow{\phi^*} & \phi^* \mathcal{L}(w, \bar{w}) \\ \downarrow & & \downarrow \\ \partial^\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu z)} - \frac{\partial \mathcal{L}}{\partial z} & \xrightarrow{\phi^*} & \partial^\mu \frac{\partial \phi^* \mathcal{L}}{\partial (\partial_\mu w)} - \frac{\partial \phi^* \mathcal{L}}{\partial w} \end{array} \quad (8.47)$$

Partition	J.s	Orbit Name	One field reduction
6=6	$(\frac{5}{2})$	\mathfrak{D}_1	NO
6=5+1	$(2, 0)$	Orbit does not exist	NO
6=4+2	$(\frac{3}{2}, \frac{1}{2})$	\mathfrak{D}_2	NO
6=3+3	$(1, 1)$	\mathfrak{D}_3	NO
6=3+2+1	$(1, \frac{1}{2}, 0)$	Orbit does not exist	NO
6=3+1+1+1	$(1, 0, 0, 0)$	Orbit does not exist	NO
6=2+2+2	$(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	\mathfrak{D}_4	YES
6=2+2+1+1	$(\frac{1}{2}, \frac{1}{2}, 0, 0)$	\mathfrak{D}_5	YES
6=2+1+1+1+1	$(\frac{1}{2}, 0, 0, 0, 0)$	\mathfrak{D}_6	YES

In the above table we have summarized the results of this simple investigation. There is a total of six orbits (up to possible further splitting in Weyl group orbits which we have not analyzed) and for each of them the corresponding immersion formulae in the $\mathcal{M}_{\text{Sp}_6}$ manifolds are those described below.

Orbit \mathfrak{D}_1 : $(j = \frac{5}{2})$.

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} = \begin{pmatrix} -6\psi^5 + 10ie^\lambda\psi^4 + 5ie^{3\lambda}\psi^2 + ie^{5\lambda} & \sqrt{5}\psi(3\psi^3 - 4ie^\lambda\psi^2 - ie^{3\lambda}) & i\sqrt{10}(i\psi + e^\lambda)\psi^2 \\ \sqrt{5}\psi(3\psi^3 - 4ie^\lambda\psi^2 - ie^{3\lambda}) & i(8i\psi^3 + 8e^\lambda\psi^2 + e^{3\lambda}) & \sqrt{2}\psi(3\psi - 2ie^\lambda) \\ i\sqrt{10}(i\psi + e^\lambda)\psi^2 & \sqrt{2}\psi(3\psi - 2ie^\lambda) & ie^\lambda - 3\psi \end{pmatrix}$$

$$w = ie^\lambda + \psi \quad (8.48)$$

The pull-back of the lagrangian is the following one:

$$\phi^* \mathcal{L} = 35 \left(\partial^\mu \psi \partial^\mu \psi e^{-2\lambda} + \partial^\mu \lambda \partial^\mu \lambda \right) - \frac{1}{4} g^2 \left(3e^{-\lambda} - \kappa \right)^2 \quad (8.49)$$

The pull-backs of the scalar field equations are inconsistent among themselves and differ from the equations derived from the pull-back of the lagrangian (8.49), hence the truncation is not consistent. No Starobinsky-like model can be obtained from this orbit.

One might wonder whether the inconsistency is due to the particularly chosen coset representative (8.42) and to the explicit form of the embedding (8.48) which turns out to be non-holomorphic. To clarify such a doubt and show that the inconsistency of the equations is an intrinsic property of the orbit, we have addressed the problem from a different view point which leads to a perfectly holomorphic embedding of the Lobachevsky plane associated with the considered orbit into the target Special Kähler manifold (7.1).

The argument is the following one. Having fixed the embedding $\mathfrak{sl}(2, \mathbb{R}) \mapsto \mathfrak{sp}(6, \mathbb{R})$ at the level of the fundamental representation **6** it is fixed also in all other representations and we can wonder what is the

branching rule of the **W**-representation $\mathbf{14}'$ such an embedding. By direct evaluation of the Casimir we obtain the following branching:

$$\mathbf{14}' \xrightarrow{\mathfrak{sl}(2, \mathbb{R})} \left(j = \frac{9}{2}\right) \oplus \left(j = \frac{3}{2}\right) \quad (8.50)$$

This means that the symplectic section (7.26) splits into the sum of two vectors, one lying in the 10-dimensional space of the first representation, the other in the 4-dimensional space of the second representation. Imposing the vanishing of the lowest spin representation introduces a set of 4 holomorphic constraints on the six coordinates z_i . By construction these constraints are $\mathfrak{sl}(2, \mathbb{R})$ invariant: therefore the sought for Lobachevsky plane certainly lies in the complex two-folds defined by the vanishing of these constraints. With a little bit of work one can further eliminate one of the two remaining complex coordinates in such a way that the ten entries of the $(j = \frac{9}{2})$ representation correspond to all the powers w^r , with $r = 0, 1, \dots, 9$ of a complex parameter w . Because of this very property w can be interpreted as the local coordinate of the sought for Lobachevsky plane embedded in the Kähler manifold (7.1) according to the specified orbit. Indeed if w transforms by fractional linear transformation under some algebra $\mathfrak{sl}(2)$, then the $2j + 1$ first powers of w provide a basis for the j -representation of that $\mathfrak{sl}(2)$. Viceversa, if a vector, which is known to transform in the j -representation of a given $\mathfrak{sl}(2)$ (up to an overall function of w), is made by linear combinations of the first $2j + 1$ powers of a coordinate w , then that w is the local coordinate on a Lobachevsky plane transitive under the action of that very $\mathfrak{sl}(2)$.

In our case the four holomorphic constraints that express the vanishing of the $j = \frac{3}{2}$ representation inside the $\mathbf{14}'$ are the following ones:

$$\begin{aligned} \sqrt{\frac{2}{7}} \left(\sqrt{5} (z_4 z_6 - z_5^2) - 2z_2 \right) &= 0 \\ \frac{8z_3 - \sqrt{10}z_4}{\sqrt{21}} &= 0 \\ \sqrt{\frac{2}{7}} \left(\sqrt{5}z_1 + 2z_3 z_5 - 2z_2 z_6 \right) &= 0 \\ \frac{-\sqrt{10}z_3^2 + 8z_4 z_3 - 8z_2 z_5 + \sqrt{10}z_1 z_6}{\sqrt{21}} &= 0 \end{aligned} \quad (8.51)$$

The explicit form of (8.51) obviously depends on the standard triple chosen as representative of the orbit, yet for whatever representative the four constraints are holomorphic. The next point consists in solving (8.51) in terms of a parameter w so that the complementary set of ten polynomials of the z_i spanning the $j = \frac{9}{2}$ representation provide all the powers of w from 0 to 9. The requested solution is given by:

$$z_1 \rightarrow \frac{3w^5}{16}, \quad z_2 \rightarrow \frac{3\sqrt{5}w^4}{16}, \quad z_3 \rightarrow \frac{1}{4}\sqrt{\frac{5}{2}}w^3, \quad z_4 \rightarrow w^3, \quad z_5 \rightarrow \frac{3w^2}{2\sqrt{2}}, \quad z_6 \rightarrow \frac{3w}{2} \quad (8.52)$$

Implementing the transformation (8.52) in the symplectic section (7.26) one finds:

$$\Omega[Z] \xrightarrow{\phi} \Omega_{\frac{9}{2}}[w] = \begin{pmatrix} \frac{3w}{\sqrt{2}} \\ \frac{w^6}{4\sqrt{2}} \\ -\frac{3w^4}{4\sqrt{2}} \\ \frac{w^9}{256\sqrt{2}} \\ -\frac{3w^7}{32\sqrt{2}} \\ -\frac{1}{16}\sqrt{\frac{5}{2}}w^6 \\ -\frac{3}{16}\sqrt{5}w^5 \\ \frac{3w^8}{128\sqrt{2}} \\ -\sqrt{2}w^3 \\ \frac{3w^5}{8\sqrt{2}} \\ \sqrt{2} \\ -\frac{3w^2}{\sqrt{2}} \\ \frac{1}{2}\sqrt{\frac{5}{2}}w^3 \\ -\frac{3}{8}\sqrt{5}w^4 \end{pmatrix} \quad (8.53)$$

which as requested contains all the powers of w and has vanishing projection on the $j = \frac{3}{2}$ representation. Calculating the Kähler potential from such a section we obtain:

$$\mathcal{K}_{\frac{9}{2}} = -\log \left(\bar{\Omega}_{\frac{9}{2}}[\bar{w}] \mathbb{C}_{14} \Omega_{\frac{9}{2}}[w] \right) = \log \left(-\frac{i}{256} (w - \bar{w})^9 \right) \quad (8.54)$$

Now the question of consistency can be readdressed in the present context. Implementing the substitution (8.52) in the six complex equations (8.45) (with for instance vanishing potential) do we obtain six consistent equations or not? The answer is no. The six equations (8.45) are inconsistent and this confirms in a holomorphic set up the same result we had previously obtained in the direct approach of eq.s (8.42-8.43). Hence the $\mathfrak{sl}(2)$ embedding of orbit \mathfrak{O}_1 leads to inconsistent truncations and has to be excluded.

Orbit \mathfrak{O}_2 : ($j_1 = \frac{3}{2}, j_2 = \frac{1}{2}$). For the second orbit, the direct approach (8.42-8.43) leads to:

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} = \begin{pmatrix} (e^\lambda - i\psi)^2 (2\psi - ie^\lambda) & 0 & \sqrt{3}\psi (\psi + ie^\lambda) \\ 0 & -\psi - ie^\lambda & 0 \\ \sqrt{3}\psi (\psi + ie^\lambda) & 0 & -2\psi - ie^\lambda \end{pmatrix} \quad (8.55)$$

$$w = ie^\lambda + \psi$$

The pull-back of the lagrangian is the following one:

$$\phi^* \mathcal{L} = 11 \left(\partial^\mu \psi \partial^\mu \psi e^{-2\lambda} + \partial^\mu \lambda \partial^\mu \lambda \right) - \frac{1}{4} g^2 \left(3 e^{-\lambda} - \kappa \right)^2 \quad (8.56)$$

Also in this case the pull-back of the scalar field equations yields an inconsistent set and there is no truncation. No Starobinsky-like model can be obtained from this orbit. In a similar way to the previous case we can discuss the same issue in a holomorphic set up. The branching rule of the $\mathbf{14}'$ representation in the considered embedding is the following one:

$$\mathbf{14}' \xrightarrow{\mathfrak{sl}(2, \mathbb{R})} \left(j = \frac{5}{2} \right) \oplus \left(j = \frac{3}{2} \right) \oplus \left(j = \frac{3}{2} \right) \quad (8.57)$$

and we can impose holomorphic constraints that suppress the two lowest spin representations ($j = \frac{3}{2}$) leaving only the top one ($j = \frac{5}{2}$) spanned by the powers of a parameter w from 0 to 5. Such a holomorphic embedding is given:

$$z_1 \frac{2w^3}{3^{3/4}}, z_2 \rightarrow 0, z_3 \rightarrow w^2, z_4 \rightarrow \frac{w}{\sqrt[4]{3}}, z_5 \rightarrow 0, z_6 \rightarrow \frac{2w}{\sqrt[4]{3}} \quad (8.58)$$

Substitution of eq.s (8.58) into the field equations (8.45) confirms that their pull-back on this surface is inconsistent.

Orbit \mathfrak{O}_3 : ($j_1 = 1, j_2 = 1$). For the third orbit, the direct approach (8.42-8.43) leads to

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} = \begin{pmatrix} -ie^{2\lambda} & -\psi^2 & -\sqrt{2}\psi \\ -\psi^2 & -i(2\psi^2 + e^{2\lambda}) & -i\sqrt{2}\psi \\ -\sqrt{2}\psi & -i\sqrt{2}\psi & -i \end{pmatrix} \\ w = ie^\lambda + \psi \quad (8.59)$$

The pull-back of the lagrangian is the following one:

$$\phi^* \mathcal{L} = 8 \left(\partial^\mu \psi \partial^\mu \psi e^{-2\lambda} + \partial^\mu \lambda \partial^\mu \lambda \right) - \frac{1}{4} g^2 \kappa^2 \quad (8.60)$$

while the pull-back of the scalar field equations is an inconsistent set. Hence this truncation is not consistent and no Starobinsky-like model can be obtained from this orbit. As in the previous two cases we can confirm the same result in a holomorphic set up, yet we consider it useless to repeat once more the same type of calculations. What is relevant to mention in view of our subsequent considerations is the branching rule of the $\mathbf{14}'$ representation under this forbidden embedding leading to inconsistent field equations::

$$\mathbf{14}' \xrightarrow{\mathfrak{sl}(2, \mathbb{R})} (j = 2) \oplus (j = 2) \oplus 4 \times (j = 0) \quad (8.61)$$

Orbit \mathfrak{O}_4 : $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = \frac{1}{2})$. For the fourth orbit, the direct approach (8.42-8.43) leads to

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} = \begin{pmatrix} ie^\lambda - \psi & 0 & 0 \\ 0 & ie^\lambda - \psi & 0 \\ 0 & 0 & ie^\lambda - \psi \end{pmatrix} \\ w = ie^\lambda - \psi \quad (8.62)$$

The pull-back of the lagrangian is the following one:

$$\phi^* \mathcal{L} = 3 \left(\partial^\mu \psi \partial^\mu \psi e^{-2\lambda} + \partial^\mu \lambda \partial^\mu \lambda \right) - g^2 \frac{1}{4} \left(3e^{-\lambda} - 2\kappa \right)^2 \quad (8.63)$$

The pull-back of the scalar field equations produces equations consistent among themselves which coincide with the equations derived from the pull-back of the lagrangian (8.63), hence the truncation is consistent. We reobtain the Starobinsky model discussed in the previous section with $q = 3$ and hence with $\alpha = 1$. In this case the consistent truncation is already produced from holomorphic constraints. Indeed equation (8.62) can be summarized as:

$$z_2 = z_3 = z_5 = 0 \quad ; \quad z_1 = z_4 = z_6 = w \quad (8.64)$$

It is interesting and important for our future consideration to mention the branching rule of the $\mathbf{14}'$ representation under this $\mathfrak{sl}(2)$ subalgebra:

$$\mathbf{14}' \xrightarrow{\mathfrak{sl}(2, \mathbb{R})} \left(j = \frac{3}{2} \right) \oplus 5 \times \left(j = \frac{1}{2} \right) \quad (8.65)$$

and the constraints (8.64) precisely are the conditions under which the five representations $(j = \frac{1}{2})$ vanish and we are left with the representation $(j = \frac{3}{2})$ duely spanned by the powers $1, w, w^2, w^3$.

Orbit \mathfrak{O}_5 : $(j_1 = \frac{1}{2}, j_2 = \frac{1}{2}, j_3 = 0)$. For the fifth orbit, the direct approach (8.42-8.43) leads to

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} = \begin{pmatrix} ie^\lambda - \psi & 0 & 0 \\ 0 & ie^\lambda - \psi & 0 \\ 0 & 0 & i \end{pmatrix} \\ w = ie^\lambda - \psi \quad (8.66)$$

The pull-back of the lagrangian is the following one:

$$\phi^* \mathcal{L} = 2 \left(\partial^\mu \psi \partial^\mu \psi e^{-2\lambda} + \partial^\mu \lambda \partial^\mu \lambda \right) - g^2 \left(e^{-\lambda} - \kappa \right)^2 \quad (8.67)$$

The pull-back of the scalar field equations yields a consistent system identical with the field equations derived from the pull-back of the lagrangian (8.71), hence the truncation is consistent. We reobtain the Starobinsky-like model discussed in the previous section with $q = 2$ and hence with $\alpha = \frac{2}{3}$.

In this, as in the previous case, the consistent truncation is produced from holomorphic constraints. Indeed equation (8.62) can be summarized as:

$$z_2 = z_3 = z_5 = 0 \quad ; \quad z_1 = z_4 = w \quad ; \quad z_6 = i \quad (8.68)$$

In this case the branching rule of the $\mathbf{14}'$ representation under the considered $\mathfrak{sl}(2)$ subalgebra is the following one:

$$\mathbf{14}' \xrightarrow{\mathfrak{sl}(2, \mathbb{R})} (j = 1) \oplus (j = 1) \oplus 2 \times \left(j = \frac{1}{2}\right) + 4 \times (j = 0) \quad (8.69)$$

and the constraint (8.68) guarantees that the singlets and the $(j = \frac{1}{2})$ representations are all set to zero.

Orbit \mathfrak{O}_6 : $(j_1 = \frac{1}{2}, j_2 = 0, j_3 = 0)$. For the sixth orbit, the direct approach (8.42-8.43) leads to

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_4 & z_5 \\ z_3 & z_5 & z_6 \end{pmatrix} = \begin{pmatrix} \psi + ie^\lambda & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{pmatrix} \\ w = ie^\lambda + \psi \quad (8.70)$$

The pull-back of the lagrangian is the following one:

$$\phi^* \mathcal{L} = \left(\partial^\mu \psi \partial^\mu \psi e^{-2\lambda} + \partial^\mu \lambda \partial^\mu \lambda \right) - \frac{1}{4} g^2 \left(e^\lambda + \kappa \right)^2 \quad (8.71)$$

The pull-back of the scalar field equations yields a consistent system coinciding with the equations derived from the pull-back of the lagrangian (8.71). So we have a consistent truncation and we reobtain the Starobinsky-like model discussed in the previous section with $q = 1$. It corresponds to $\alpha = \frac{1}{3}$. Equation (8.70) can be summarized as:

$$z_2 = z_3 = z_5 = 0 \quad ; \quad z_1 = w \quad ; \quad z_4 = z_6 = i \quad (8.72)$$

The branching of the $\mathbf{14}'$ dimensional representation under this $\mathfrak{sl}(2)$ subalgebra is the following one:

$$\mathbf{14}' \xrightarrow{\mathfrak{sl}(2, \mathbb{R})} 5 \times \left(j = \frac{1}{2}\right) \oplus 4 \times (j = 0) \quad (8.73)$$

8.5.1 CONCLUSION OF THE ABOVE DISCUSSION

This concludes our preliminary study of the orbits and shows that the embedded Starobinsky-like models described in section 8.4 exhaust the list of possible embeddings, the values of $\alpha = 1, \frac{2}{3}, \frac{1}{3}$ being, apparently

the only admissible ones. Next let us observe that the branching rules of the $\mathbf{14}'$ dimensional representation which lead to consistent truncations, namely, (8.65,8.69,8.73) are the only possible ones that we can obtain by embedding:

$$\mathfrak{sl}(2) \mapsto \mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2) \quad (8.74)$$

if the considered $\mathbf{14}'$ representation of $\mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2)$ is the following one:

$$\mathbf{14}' = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \oplus \left(\frac{1}{2}, 0, 0\right) \oplus \left(0, \frac{1}{2}, 0\right) \oplus \left(0, 0, \frac{1}{2}\right) \quad (8.75)$$

This has a profound meaning. It implies that the only consistent truncations occur when the $\mathfrak{sl}(2)$ Lie algebra is embedded in the *sub-Tits-Satake* Lie algebra, which as we discuss in the conclusive part is universal for all $\mathcal{N} = 2$ models. This allows us to make the bold statement that the only values of α one can obtain from the gauging of hypermultiplet isometries in any supergravity theory based on symmetric manifolds is just $\alpha = 1, \frac{2}{3}, \frac{1}{3}$.

Part V

CONCLUSIONS AND PERSPECTIVES

Considering the structure of the c -map and the results relative to the inclusion of Starobinsky like potentials that we have concretely obtained in the case of the $\mathrm{Sp}(6, \mathbb{R})$ -model, we interpret them within the framework provided by Tits-Satake subalgebras and Tits Satake universality classes. This allows us to advocate that the mechanism underlying the generation of such cosmological potentials is universal and the prediction on the possible values of α equally general. Actually the analysis we are going to present in the next section supports the following general conclusion: the cosmological potentials classified in [41], that follow from the gauging of a generator respectively elliptic, hyperbolic or parabolic of a constant curvature Kähler surface admit a universal uplifting to all $\mathcal{N} = 2$ models based on symmetric spaces for the hypermultiplets, provided the curvature of the Kähler surface is duely quantized. Indeed there exist consistent one-field truncation only if the gauging occurs inside the universal *sub-Tits-Satake* Lie algebra.

9 THE TITS SATAKE PROJECTION

In most cases of lower supersymmetry, neither the algebra $\mathbb{U}_{\mathcal{K}}$ nor the algebra $\mathbb{U}_{\mathcal{QM}}$ are **maximally split**. In short this means that the non-compact rank $r_{nc} < r$ is less than the rank of \mathbb{U} , namely not all the Cartan generators are non-compact. Rigorously r_{nc} is defined as follows:

$$r_{nc} = \mathrm{rank}(\mathbb{U}/\mathbb{H}) \equiv \dim \mathcal{H}^{n.c.} \quad ; \quad \mathcal{H}^{n.c.} \equiv \mathrm{CSA}_{\mathbb{U}(\mathbb{C})} \bigcap \mathbb{K} \quad (9.1)$$

When this happens it means that the structure of both black-hole-like and cosmological-like solutions of supergravity is effectively determined by a *maximally split subalgebra* $\mathbb{U}^{TS} \subset \mathbb{U}$ named the *Tits Satake* subalgebra of \mathbb{U} , whose rank is equal to r_{nc} . Effectively determined does not mean that solutions of the big system coincide with those of the smaller system rather it means that the former can be obtained from the latter by means of rotations of a compact subgroup $G_{\mathrm{paint}} \subset \mathbb{U}$ of the big group which we name the *paint group*, for whose precise definition we refer the reader to [3]. Here we just emphasize few important facts, relevant for our goals. To this effect we recall that the Tits Satake algebra is obtained from the original algebra via a projection of the root system of \mathbb{U} onto the subspace orthogonal to the compact part of the Cartan subalgebra of \mathbb{U}^{TS} :

$$\Pi^{TS} \quad ; \quad \Delta_{\mathbb{U}} \mapsto \overline{\Delta}_{\mathbb{U}^{TS}} \quad (9.2)$$

In euclidian geometry $\overline{\Delta}_{\mathbb{U}^{TS}}$ is just a collection of vectors in r_{nc} dimensions; a priori there is no reason why it should be the root system of another Lie algebra. Yet in almost all cases, $\overline{\Delta}_{\mathbb{U}^{TS}}$ turns out to be a Lie algebra root system and the maximal split Lie algebra corresponding to it, \mathbb{U}^{TS} , is, by definition, the Tits Satake subalgebra of the original non maximally split Lie algebra: $\mathbb{U}^{TS} \subset \mathbb{U}$. Such algebras \mathbb{U} are called *non-exotic*. The *exotic* non compact algebras are those for which the system $\overline{\Delta}_{\mathbb{U}^{TS}}$ is not an admissible root system. In such cases there is no Tits Satake subalgebra \mathbb{U}^{TS} . Exotic subalgebra are very few and in supergravity they appear only in three instances that display additional peculiarities relevant for the black-hole and cosmological solutions. As for $\mathcal{N} = 2$ theories the only exotic homogeneous symmetric Special Kähler manifolds are those of the Minimal Coupling series discussed in section 5. Exotic are also the Quaternionic Kähler manifolds in the c -map image of the former.

For the non exotic models we have that the decomposition (4.46) commutes with the projection, namely:

$$\begin{aligned} \text{adj}(\mathbb{U}_{\mathcal{QM}}) &= \text{adj}(\mathbb{U}_{\mathcal{SK}}) \oplus \text{adj}(\mathfrak{sl}(2, \mathbb{R})_E) \oplus \mathbf{W}_{(2,W)} \\ &\Downarrow \\ \text{adj}(\mathbb{U}_{\mathcal{QM}}^{TS}) &= \text{adj}(\mathbb{U}_{\mathcal{SK}}^{TS}) \oplus \text{adj}(\mathfrak{sl}(2, \mathbb{R})_E) \oplus \mathbf{W}_{(2,W^{TS})} \end{aligned} \quad (9.3)$$

In other words the projection leaves the A_1 Ehlers subalgebra untouched and has a non trivial effect only on the algebra $\mathbb{U}_{\mathcal{SK}}$. Furthermore the image under the projection of the highest root of \mathbb{U} is the highest root of \mathbb{U}^{TS} :

$$\Pi^{TS} : \psi \rightarrow \psi^{TS} \quad (9.4)$$

The reason why the Tits Satake projection is relevant to us is that the classification of nilpotent orbits (standard triples) and hence of abelian gaugings depends only on the Tits Satake subalgebra and therefore is universal for all members of the same Tits Satake universality class. By this name we mean all algebras that share the same Tits Satake projection. A similar property was extensively used in [4] in the discussion of extremal black-hole solutions. Indeed the classification of extremal black-holes also boils down to the classification of nilpotent orbits, so that the mathematical problem at stake is just the same.

Having clarified these points we can proceed with the classification of homogeneous symmetric spaces relevant to $\mathcal{N} = 2$ supergravity either in the vector multiplet sector (Special Kähler) or in the hypermultiplet sector (Quaternionic Kähler). These spaces are listed in table 2. In table 2 we have also listed the Paint groups and the subpaint groups. These latter are always compact and their different structures is what distinguishes the different elements belonging to the same class. As it was shown in [3], these groups are dimensional reduction invariant and therefore c -map invariant, namely they are the same in $\mathbb{U}_{\mathcal{SK}}$ and in $\mathbb{U}_{\mathcal{QM}}$. Hence the representation \mathbf{W} which, as we have seen, hosts the symplectic section of Special Geometry and regulates, by means of its branching, the existence or non existence of consistent truncations, can be decomposed with respect to the Tits Satake subalgebra and the Paint group revealing a regularity structure inside each Tits Satake universality class which is what allows us to draw general conclusions and make universal predictions. In the case of black-holes the same Tits-Satake decomposition of the \mathbf{W} -representation is at the heart of the classification of *charge orbits* of the hole, as we extensively discussed in [4].

10 TITS SATAKE UNIVERSALITY CLASSES AND THE EMBEDDING OF STAROBINSKY-LIKE MODELS

In the present section we consider the decomposition of the \mathbf{W} -representations with respect to Tits-Satake subalgebras and Paint groups for all the non-exotic models.

In [3] the *paint algebra* was defined as the algebra of external automorphisms of the solvable Lie algebra $\text{Solv}_{\mathcal{M}}$ generating the non-compact symmetric space: $\mathcal{M} = \mathbb{U}/\mathbb{H}$, namely

$$\mathbb{G}_{\text{paint}} = \text{Aut}_{\text{Ext}} [\text{Solv}_{\mathcal{M}}]. \quad (10.1)$$

where:

$$\text{Aut}_{\text{Ext}} [\text{Solv}_{\mathcal{M}}] \equiv \frac{\text{Aut} [\text{Solv}_{\mathcal{M}}]}{\text{Solv}_{\mathcal{M}}}, \quad (10.2)$$

Given the paint algebra $\mathbb{G}_{\text{paint}} \subset \mathbb{U}$ and the Tits Satake subalgebra $\mathbb{G}_{\text{TS}} \subset \mathbb{U}$, whose construction we have briefly recalled above, following [3] one introduces the *sub Tits Satake* and *sub paint* algebras as the

#	TS SK_{TS}	TS QM_{TS}	coset SK	coset QM	Paint Group	subP Group	susy
1	$\frac{SU(1,1)}{U(1)}$	$\frac{G_{2(2)}}{SU(2) \times SU(2)}$	$\frac{SU(1,1)}{U(1)}$	$\frac{G_{2(2)}}{SU(2) \times SU(2)}$	1	1	$\mathcal{N} = 2$ $n=1$
2	$\frac{Sp(6, \mathbb{R})}{SU(3) \times U(1)}$	$\frac{F_{4(4)}}{Sp(6, \mathbb{R}) \times SL(2, \mathbb{R})}$	$\frac{Sp(6, \mathbb{R})}{SU(3) \times U(1)}$	$\frac{F_{4(4)}}{USp(6) \times SU(2)}$	1	1	$\mathcal{N} = 2$ $n = 6$
3			$\frac{SU(3,3)}{SU(3) \times SU(3) \times U(1)}$	$\frac{E_{6(2)}}{SU(6) \times SU(2)}$	$SO(2) \times SO(2)$	1	$\mathcal{N} = 2$ $n = 9$
4			$\frac{SO^*(12)}{SU(6) \times U(1)}$	$\frac{E_{7(-5)}}{SO(12) \times SU(2)}$	$SO(3) \times SO(3)$ $\times SO(3)$	$SO(3)_d$	$\mathcal{N} = 6$ $\mathcal{N} = 2$ $n=15$
5			$\frac{E_{7(-25)}}{E_{6(-78)} \times U(1)}$	$\frac{E_{8(-24)}}{E_{7(-133)} \times SU(2)}$	$SO(8)$	$G_{2(2)}$	$\mathcal{N} = 2$ $n = 27$
6	$\frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SO(2,1)}{SO(2)}$	$\frac{SO(4,3)}{SO(4) \times SO(3)}$	$\frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SO(2,1)}{SO(2)}$	$\frac{SO(4,3)}{SO(4) \times SO(3)}$	1	1	$\mathcal{N} = 2$ $n=2$
7	$\frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SO(2,2)}{SO(2) \times SO(2)}$	$\frac{SO(4,4)}{SO(4) \times SO(4)}$	$\frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SO(2,2)}{SO(2) \times SO(2)}$	$\frac{SO(4,4)}{SO(4) \times SO(4)}$	1	1	$\mathcal{N} = 2$ $n=3$
8	$\frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SO(2,3)}{SO(2) \times SO(3)}$	$\frac{SO(4,5)}{SO(4) \times SO(5)}$	$\frac{SL(2, \mathbb{R})}{SO(2)} \times \frac{SO(2,2+p)}{SO(2) \times SO(2+p)}$	$\frac{SO(4,4+p)}{SO(4) \times SO(4+p)}$	$SO(p)$	$SO(p-1)$	$\mathcal{N} = 2$ $n=3+p$
exot	bc_1	bc_2	$\frac{SU(p+1,1)}{SU(p+1) \times U(1)}$	$\frac{SU(p+2,2)}{SU(p+1,1) \times SL(2, \mathbb{R})_{h^*}}$	$U(1) \times U(1) \times U(p)$	$U(p-1)$	$\mathcal{N} = 2$ $n=p+1$

Table 2: The first eight rows of this table list the *non-exotic* homogenous symmetric Special Kähler manifolds. They are displayed together with their Quaternionic Kähler c -map images and are organized in five Tits Satake universality classes. Non exotic means that the Tits Satake projection of the root system is a standard Lie Algebra root system. Of the five universality classes three contain only one maximally split element, one contains four elements filling the Tits magic square, while the last class contains an infinite number of elements. Within each class the models are distinguished by the different structure of the Paint Group and of its subPaint subgroup. The Paint group is a c -map invariant. It is the same in the Special Kähler and in the Quaternionic Kähler case. The last line displays the unique family of exotic Special Kähler symmetric spaces for which the Tits Satake projection of the root system is not a root system. They correspond to the Minimal Coupling Models discussed in the main text. Their c -map images of the exotic models are also exotic. Notwithstanding this anomaly the concept of Paint Group, according to its definition as group of external automorphisms of the solvable Lie algebra generating the non compact coset manifold still exists. The Paint group is the same for the Kähler and the quaternionic Kähler case as in the non exotic cases.

centralizers of the paint algebra and of the Tits Satake algebra, respectively. In other words we have:

$$\mathfrak{s} \in \mathbb{G}_{\text{subTS}} \subset \mathbb{G}_{\text{TS}} \subset \mathbb{U} \quad \Leftrightarrow \quad [\mathfrak{s}, \mathbb{G}_{\text{paint}}] = 0 \quad (10.3)$$

and

$$\mathfrak{t} \in \mathbb{G}_{\text{subpaint}} \subset \mathbb{G}_{\text{paint}} \subset \mathbb{U} \quad \Leftrightarrow \quad [\mathfrak{t}, \mathbb{G}_{\text{TS}}] = 0 \quad (10.4)$$

A very important property of the paint and subpaint algebras is that they are conserved int the c -map,

namely they are the same for $\mathbb{U}_{\mathcal{SK}}$ and $\mathbb{U}_{\mathcal{QM}}$.

In the next lines we analyze the decomposition of the **W**-representations with respect to these subalgebras for each Tits Satake universality class of non maximally split models. In the case of maximally split models there is no paint algebra and there is nothing with respect to which to decompose.

10.1 UNIVERSALITY CLASS $\mathfrak{sp}(6, \mathbb{R}) \Rightarrow \mathfrak{f}_{4(4)}$

In this case the sub Tits Satake Lie algebra is

$$\mathbb{G}_{\text{subTS}} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{sp}(6, \mathbb{R}) = \mathbb{G}_{\text{TS}} \quad (10.5)$$

and the **W**-representation of the maximally split model decomposes as follows:

$$\mathbf{14}' \xrightarrow{\mathbb{G}_{\text{subTS}}} (\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{2}, \mathbf{2}) \quad (10.6)$$

This decomposition combines in the following way with the paint group representations in the various models belonging to the same universality class.

10.1.1 $\mathfrak{su}(3, 3)$ MODEL

For this case the paint algebra is

$$\mathbb{G}_{\text{paint}} = \mathfrak{so}(2) \oplus \mathfrak{so}(2) \quad (10.7)$$

and the **W**-representation is the **20** dimensional of $\mathfrak{su}(3, 3)$ corresponding to an antisymmetric tensor with a reality condition of the form:

$$t_{\alpha\beta\gamma}^* = \frac{1}{3!} \epsilon_{\alpha\beta\gamma\delta\eta\theta} t_{\delta\eta\theta} \quad (10.8)$$

The decomposition of this representation with respect to the Lie algebra $\mathbb{G}_{\text{paint}} \oplus \mathbb{G}_{\text{subTS}}$ is the following one:

$$\mathbf{20} \xrightarrow{\mathbb{G}_{\text{paint}} \oplus \mathbb{G}_{\text{subTS}}} (2, q_1 | \mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (2, q_2 | \mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (2, q_3 | \mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (1, 0 | \mathbf{2}, \mathbf{2}, \mathbf{2}) \quad (10.9)$$

where $(2, q)$ means a doublet of $\mathfrak{so}(2) \oplus \mathfrak{so}(2)$ with a certain grading q with respect to the generators, while $(1, 0)$ means the singlet that has 0 grading with respect to both generators. The subpaint algebra in this case is $\mathbb{G}_{\text{subpaint}} = 0$ and the decomposition of the same **W**-representation with respect to $\mathbb{G}_{\text{subpaint}} \oplus \mathbb{G}_{\text{TS}}$ is:

$$\mathbf{20} \xrightarrow{\mathbb{G}_{\text{subpaint}} \oplus \mathbb{G}_{\text{TS}}} \mathbf{6} \oplus \mathbf{14} \quad (10.10)$$

This follows from the decomposition of the **6** of $\mathfrak{sp}(6, \mathbb{R})$ with respect to the sub Tits Satake algebra (10.5):

$$\mathbf{6} \xrightarrow{\mathbb{G}_{\text{subTS}}} (\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}, \mathbf{2}) \quad (10.11)$$

10.1.2 $\mathfrak{so}^*(12)$ MODEL

For this case the paint algebra is

$$\mathbb{G}_{\text{paint}} = \mathfrak{so}(3) \oplus \mathfrak{so}(3) \oplus \mathfrak{so}(3) \quad (10.12)$$

and the **W**-representation is the $\mathbf{32}_s$ dimensional spinorial representation of $\mathfrak{so}^*(12)$. The decomposition of this representation with respect to the Lie algebra $\mathbb{G}_{\text{paint}} \oplus \mathbb{G}_{\text{subTS}}$ is the following one:

$$\mathbf{32}_s \xrightarrow{\mathbb{G}_{\text{paint}} \oplus \mathbb{G}_{\text{subTS}}} (\underline{2}, \underline{2}, \underline{1} | \mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\underline{2}, \underline{1}, \underline{2} | \mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\underline{1}, \underline{1}, \underline{2} | \mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\underline{1}, \underline{1}, \underline{1} | \mathbf{2}, \mathbf{2}, \mathbf{2}) \quad (10.13)$$

where $\underline{2}$ means the doublet spinor representation of $\mathfrak{so}(3)$. The subpaint algebra in this case is $\mathbb{G}_{\text{paint}} = \mathfrak{so}(3)_{\text{diag}}$ and the decomposition of the same \mathbf{W} -representation with respect to $\mathbb{G}_{\text{subpaint}} \oplus \mathbb{G}_{\text{TS}}$ is:

$$\mathbf{32}_s \xrightarrow{\mathbb{G}_{\text{TS}} \oplus \mathbb{G}_{\text{subpaint}}} (\mathbf{6}|\underline{3}) \oplus (\mathbf{14}'|\underline{1}) \quad (10.14)$$

This follows from the decomposition of the product $\underline{2} \times \underline{2}$ of $\mathfrak{so}(3)_{\text{diag}}$ times the Tits Satake algebra (10.5):

$$\underline{2} \times \underline{2} = \underline{3} \oplus \underline{1} \quad (10.15)$$

10.1.3 $\mathfrak{e}_{7(-25)}$ MODEL

For this case the paint algebra is

$$\mathbb{G}_{\text{paint}} = \mathfrak{so}(8) \quad (10.16)$$

and the \mathbf{W} -representation is the fundamental $\mathbf{56}$ dimensional representation of $\mathfrak{e}_{7(-25)}$. The decomposition of this representation with respect to the Lie algebra $\mathbb{G}_{\text{paint}} \oplus \mathbb{G}_{\text{subTS}}$ is the following one:

$$\mathbf{56} \xrightarrow{\mathbb{G}_{\text{paint}} \oplus \mathbb{G}_{\text{subTS}}} (\mathbf{8}_v|\mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathbf{8}_s|\mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathbf{8}_c|\mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1}|\mathbf{2}, \mathbf{2}, \mathbf{2}) \quad (10.17)$$

where $\mathbf{8}_{v,s,c}$ are the three inequivalent eight-dimensional representations of $\mathfrak{so}(8)$, the vector, the spinor and the conjugate spinor. The subpaint algebra in this case is $\mathbb{G}_{\text{paint}} = \mathfrak{g}_{2(-14)}$ with respect to which all three 8-dimensional representations of $\mathfrak{so}(8)$ branch as follows:

$$\mathbf{8}_{v,s,c} \xrightarrow{\mathfrak{g}_{2(-14)}} \mathbf{7} \oplus \mathbf{1} \quad (10.18)$$

In view of this the decomposition of the same \mathbf{W} -representation with respect to $\mathbb{G}_{\text{subpaint}} \oplus \mathbb{G}_{\text{TS}}$ is:

$$\mathbf{56} \xrightarrow{\mathbb{G}_{\text{TS}} \oplus \mathbb{G}_{\text{subpaint}}} (\mathbf{6}|\mathbf{7}) \oplus (\mathbf{14}'|\mathbf{1}) \quad (10.19)$$

10.2 UNIVERSALITY CLASS $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2, 3) \Rightarrow \mathfrak{so}(4, 5)$

This case corresponds to one of the possible infinite families of $\mathcal{N} = 2$ theories with a symmetric homogeneous special Kähler manifold and a number of vector multiplets larger than three ($n = 3 + p$). The other infinite family corresponds instead to one of the three exotic models.

The generic element of this infinite class corresponds to the following algebras:

$$\begin{aligned} \mathbb{U}_{\mathcal{SK}} &= \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2, 2 + p) \\ \mathbb{U}_{\mathcal{QM}} &= \mathfrak{so}(4, 4 + p) \end{aligned} \quad (10.20)$$

In this case the sub Tits Satake algebra is:

$$\mathbb{G}_{\text{subTS}} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \simeq \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2, 2) \subset \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2, 3) = \mathbb{G}_{\text{TS}} \quad (10.21)$$

an the paint and subpaint algebras are as follows:

$$\begin{aligned} \mathbb{G}_{\text{paint}} &= \mathfrak{so}(p) \\ \mathbb{G}_{\text{subpaint}} &= \mathfrak{so}(p - 1) \end{aligned} \quad (10.22)$$

The symplectic \mathbf{W} representation of $\mathbb{U}_{\mathcal{SK}}$ is the tensor product of the fundamental representation of $\mathfrak{sl}(2)$ with the fundamental vector representation of $\mathfrak{so}(2, 2 + p)$, namely

$$\mathbf{W} = (\mathbf{2}|4 + p) \quad ; \quad \dim \mathbf{W} = 8 + 2p \quad (10.23)$$

The decomposition of this representation with respect to $\mathbb{G}_{\text{subTS}} \oplus \mathbb{G}_{\text{subpaint}}$ is the following one:

$$\mathbf{W} \xrightarrow{\mathbb{G}_{\text{subTS}} \oplus \mathbb{G}_{\text{subpaint}}} (\mathbf{2}, \mathbf{2}, \mathbf{2}|\mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}|\mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}|p - 1) \quad (10.24)$$

where $\mathbf{2}, \mathbf{2}, \mathbf{2}$ denotes the tensor product of the three fundamental representations of $\mathfrak{sl}(2, \mathbb{R})^3$. Similarly $\mathbf{2}, \mathbf{1}, \mathbf{1}$ denotes the doublet of the first $\mathfrak{sl}(2, \mathbb{R})$ tensored with the singlets of the following two $\mathfrak{sl}(2, \mathbb{R})$ algebras. The representations appearing in (10.24) can be grouped in order to reconstruct full representations either of the complete Tits Satake or of the complete paint algebras. In this way one obtains:

$$\begin{aligned} \mathbf{W} &\xrightarrow{\mathbb{G}_{\text{subTS}} \oplus \mathbb{G}_{\text{paint}}} (\mathbf{2}, \mathbf{2}, \mathbf{2}|\mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}, \mathbf{1}|p + 1) \\ \mathbf{W} &\xrightarrow{\mathbb{G}_{\text{TS}} \oplus \mathbb{G}_{\text{subpaint}}} (\mathbf{2}, \mathbf{5}|\mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}|p - 1) \end{aligned} \quad (10.25)$$

10.3 \mathbf{W} -REPRESENTATIONS OF THE MAXIMALLY SPLIT NON EXOTIC MODELS

In the previous subsections we have analyzed the Tits-Satake decomposition of the \mathbf{W} -representation for all those models that are non maximally split. The remaining models are the maximally split ones for which there is no paint algebra and the Tits Satake projection is the identity map. There are essentially four type of models:

1. The $\text{SU}(1, 1)$ non exotic model where the \mathbf{W} -representation is the $j = \frac{3}{2}$ of $\mathfrak{so}(1, 2) \sim \mathfrak{su}(1, 1)$
2. The $\text{Sp}(6, \mathbb{R})$ model where the \mathbf{W} -representation is the $\mathbf{14}'$ (antisymmetric symplectic traceless three-tensor).
3. The models $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(q, q)$ where the \mathbf{W} -representation is the $(2, 2q)$, namely the tensor product of the two fundamentals.
4. The models $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(q, q + 1)$ where the \mathbf{W} -representation is the $(2, 2q + 1)$, namely the tensor product of the two fundamentals.

Therefore, for the above maximally split models, we need the classification of $\mathbb{U}_{\mathcal{SK}}$ orbits in the mentioned \mathbf{W} -representations. Actually such orbits are sufficient also for the non maximally split models. Indeed each of the above 4-models correspond to one Tits Satake universality class and, within each universality class, the only relevant part of the \mathbf{W} -representation is the subpaint group singlet which is universal for all members of the class. This is precisely what we verified in the previous subsections.

For instance for all members of the universality class of $\text{Sp}(6, \mathbb{R})$, the \mathbf{W} -representation splits as follows with respect to the subalgebra $\mathfrak{sp}(6, \mathbb{R}) \oplus \mathbb{G}_{\text{subpaint}}$:

$$\mathbf{W} \xrightarrow{\mathfrak{sp}(6, \mathbb{R}) \oplus \mathbb{G}_{\text{subpaint}}} (\mathbf{6} | \mathcal{D}_{\text{subpaint}}) + (\mathbf{14}' | \mathbf{1}_{\text{subpaint}}) \quad (10.26)$$

where the representation $\mathcal{D}_{\text{subpaint}}$ is the following one for the three non-maximally split members of the class:

$$\mathcal{D}_{\text{subpaint}} = \begin{cases} \mathbf{1} & \text{of } \mathbf{1} & \text{for the } \mathfrak{su}(3, 3) - \text{model} \\ \mathbf{3} & \text{of } \mathfrak{so}(3) & \text{for the } \mathfrak{so}^*(12) - \text{model} \\ \mathbf{7} & \text{of } \mathfrak{g}_{2(-14)} & \text{for the } \mathfrak{e}_{7(-25)} - \text{model} \end{cases} \quad (10.27)$$

Clearly the condition:

$$(\mathbf{6} | \mathcal{D}_{\text{subpaint}}) = 0 \quad (10.28)$$

imposed on a vector in the \mathbf{W} -representation breaks the group $U_{S\mathcal{K}}$ to its Tits Satake subgroup. The key point is that each \mathbf{W} -orbit of the big group $U_{S\mathcal{K}}$ crosses the locus (10.28) so that the classification of $\text{Sp}(6, \mathbb{R})$ orbits in the $\mathbf{14}'$ -representation exhausts the classification of \mathbf{W} -orbits for all members of the universality class.

In order to prove that the gauge (10.28) is always reachable it suffices to show that the representation $(\mathbf{6} | \mathcal{D}_{\text{subpaint}})$ always appears at least once in the decomposition of the Lie algebra $\mathbb{U}_{S\mathcal{K}}$ with respect to the subalgebra $\mathfrak{sp}(6, \mathbb{R}) \oplus \mathbb{G}_{\text{subpaint}}$. The corresponding parameters of the big group can be used to set to zero the projection of the \mathbf{W} -vector onto $(\mathbf{6} | \mathcal{D}_{\text{subpaint}})$.

The required condition is easily verified since we have:

$$\begin{array}{ccc}
\underbrace{\text{adj } \mathfrak{su}(3, 3)}_{\mathbf{35}} & \xrightarrow{\mathfrak{sp}(6, \mathbb{R})} & \underbrace{\text{adj } \mathfrak{sp}(6, \mathbb{R})}_{\mathbf{21}} \oplus \mathbf{6} \oplus \mathbf{6} \oplus \mathbf{1} \oplus \mathbf{1} \\
\underbrace{\text{adj } \mathfrak{so}^*(12)}_{\mathbf{66}} & \xrightarrow{\mathfrak{sp}(6, \mathbb{R}) \oplus \mathfrak{so}(3)} & \underbrace{\text{adj } \mathfrak{sp}(6, \mathbb{R})}_{\mathbf{21}} \oplus \underbrace{\text{adj } \mathfrak{so}(3)}_{\mathbf{3}} \oplus (\mathbf{6}, \mathbf{3}) \oplus (\mathbf{6}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \mathbf{3}) \\
\underbrace{\text{adj } \mathfrak{e}_{7(-25)}}_{\mathbf{133}} & \xrightarrow{\mathfrak{sp}(6, \mathbb{R}) \oplus \mathfrak{g}_{2(-14)}} & \underbrace{\text{adj } \mathfrak{sp}(6, \mathbb{R})}_{\mathbf{21}} \oplus \underbrace{\text{adj } \mathfrak{g}_{2(-14)}}_{\mathbf{14}} \oplus (\mathbf{6}, \mathbf{7}) \oplus (\mathbf{6}, \mathbf{7}) \oplus (\mathbf{1}, \mathbf{7}) \oplus (\mathbf{1}, \mathbf{7})
\end{array} \quad (10.29)$$

The reader cannot avoid being impressed by the striking similarity of the above decompositions which encode the very essence of Tits Satake universality. Indeed the representations of the common Tits Satake subalgebra appearing in the decomposition of the adjoint are the same for all members of the class. They are simply uniformly assigned to the fundamental representation of the subpaint algebra which is different in the three cases. The representation $(\mathbf{6} | \mathcal{D}_{\text{subpaint}})$ appears twice in these decompositions and can be used to reach the gauge (10.28) as we claimed above.

HOLOMORPHIC CONSISTENT TRUNCATIONS The next point to remark is that the condition (10.28) has another important interpretation if applied to the holomorphic section of special geometry. The key point is the following numerical identity valid for all members of the universality class:

$$\dim \frac{U_{S\mathcal{K}}}{H_{S\mathcal{K}}} = \dim \frac{U_{S\mathcal{K}}^{TS}}{H_{S\mathcal{K}}^{TS}} \oplus 6 \times \dim \mathcal{D}_{\text{subpaint}} \quad (10.30)$$

This means that if we decompose the symplectic section of the big group according to the Tits-Satake subalgebra and we impose on it the condition (10.28) we just obtain the right number of holomorphic constraints to project onto the submanifold $\frac{U_{S\mathcal{K}}}{H_{S\mathcal{K}}}$. At the level of field equations this is certainly a consistent truncation, since we project onto the singlets of the subpaint group.

On the other hand if we decompose the \mathbf{W} -representation with respect to the sub-Tits-Satake subalgebra $\mathfrak{sl}(2) \times \mathfrak{sl}(2) \times \mathfrak{sl}(2)$ we have the branching rule:

$$\mathbf{W} \rightarrow (\mathcal{D}_1 | \mathbf{2}, \mathbf{1}, \mathbf{1}) \oplus (\mathcal{D}_2 | \mathbf{1}, \mathbf{2}, \mathbf{1}) \oplus (\mathcal{D}_3 | \mathbf{1}, \mathbf{1}, \mathbf{2}) \oplus (\mathbf{1} | \mathbf{2}, \mathbf{2}, \mathbf{2}) \quad (10.31)$$

where $\mathcal{D}_{1,2,3}$ are three suitable representations of the Paint Group. Imposing on the symplectic section of the big model the constraints:

$$\begin{aligned}(\mathcal{D}_1|\mathbf{2}, \mathbf{1}, \mathbf{1}) &= 0 \\(\mathcal{D}_2|\mathbf{1}, \mathbf{2}, \mathbf{1}) &= 0 \\(\mathcal{D}_3|\mathbf{1}, \mathbf{1}, \mathbf{2}) &= 0\end{aligned}\tag{10.32}$$

yields precisely the correct number of holomorphic constraints that restrict the considered Special Kähler manifold to the Special Kähler manifold of the STU-model namely to $\left(\frac{\mathrm{SL}(2, \mathbb{R})}{\mathrm{SO}(2)}\right)^3$. This follows from the numerical identity true for all members of the universality class:

$$\dim \frac{\mathrm{U}_{\mathcal{SK}}}{\mathrm{H}_{\mathcal{SK}}} = \sum_{i=1}^3 2 \times \dim \mathcal{D}_i + 6\tag{10.33}$$

The reason why the truncation to the STU-model is always a consistent truncation at the level of field equations is obvious in this set up. It corresponds to the truncation to the Paint Group singlets.

W-REPRESENTATIONS FOR THE REMAINING MODELS For the models of type $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(q, q + p)$ having $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(q, q + 1)$ as Tits Satake subalgebra and $\mathfrak{so}(p - 1)$ as subpaint algebra the decomposition of the **W**-representation is the following one:

$$\mathbf{W} = (\mathbf{2}, \mathbf{2q} + \mathbf{p}) \xrightarrow{\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(q, q+1) \oplus \mathfrak{so}(p-1)} (\mathbf{2}, \mathbf{2q} + \mathbf{1}|\mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}|\mathbf{p} - \mathbf{1})\tag{10.34}$$

and the question is whether each $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(q, q + p)$ orbit in the $(\mathbf{2}, \mathbf{2q} + \mathbf{p})$ representation intersects the $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(q, q + 1) \oplus \mathfrak{so}(p - 1)$ -invariant locus:

$$(\mathbf{2}, \mathbf{1}|\mathbf{p} - \mathbf{1}) = 0\tag{10.35}$$

The answer is yes since we always have enough parameters in the coset

$$\frac{\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(q, q + p)}{\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SO}(q, q + 1) \times \mathrm{SO}(p - 1)}\tag{10.36}$$

to reach the desired gauge (10.35). Indeed let us observe the decomposition:

$$\mathrm{adj} [\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(q, q + p)] = \mathrm{adj} [\mathfrak{sl}(2, \mathbb{R})] \oplus \mathrm{adj} [\mathfrak{so}(q, q + 1)] \oplus \mathrm{adj} [\mathfrak{so}(p - 1)] \oplus (\mathbf{1}, \mathbf{2q} + \mathbf{1}|\mathbf{p} - \mathbf{1})\tag{10.37}$$

The $2q + 1$ vectors of $\mathfrak{so}(p - 1)$ appearing in (10.37) are certainly sufficient to set to zero the 2 vectors of $\mathfrak{so}(p - 1)$ appearing in **W**.

Relevant for the case of $\mathcal{N} = 2$ supersymmetry is the value $q = 2$ and in this case the sub-Tits-Satake Lie algebra is :

$$\mathbb{G}_{\mathrm{subTS}} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(2, 2) = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})\tag{10.38}$$

namely it is once again the Lie algebra of the STU-model. Reduction to the STU-model is consistent for the same reason as in the other universality classes: it corresponds to truncation to Paint Group singlets.

10.4 GAUGINGS WITH CONSISTENT ONE-FIELD TRUNCATIONS

On the basis of the analysis presented in the previous section we arrive at the following conclusion. By gauging a nilpotent element of the isometry subalgebra of \mathcal{SK} inside \mathcal{QM} we generate a potential. The structure of the theory depends on the nilpotent orbit, namely on the embedding of an $\mathfrak{sl}(2)$ Lie algebra in $\mathbb{U}_{\mathcal{SK}}$ and there are many ways of doing this (the orbits), yet the gauged theory will admit a one-field truncation if and only if the $\mathfrak{sl}(2)$ is embedded into the sub Tits Satake Lie algebra:

$$\mathfrak{sl}(2) \mapsto \mathbb{G}_{\text{subTS}} \subset \mathbb{U}_{\mathcal{SK}} \quad (10.39)$$

There are only three different embeddings of $\mathfrak{sl}(2)$ into $(\mathfrak{sl}(2))^3$ and these correspond to the three admissible values $\alpha = 1, \frac{2}{3}, \frac{1}{3}$ in the Starobinsky-like model.

11 CONCLUSIONS

In this paper we have analyzed in detail the structure of the c -map from Special Kähler manifolds to Quaternionic Kähler manifolds in connection with the abelian gaugings of hypermultiplet isometries in $\mathcal{N} = 2$ supergravity and the generation of effective one field potentials that might describe inflaton dynamics.

The motivations of such a study have been put forward in the introduction. Here we try to summarize the main results we have obtained and the perspectives for future investigations that have emerged.

RESULTS

I The c -map description of Quaternionic Kähler manifolds \mathcal{QM}_{4n+4} allows to distribute the isometries of \mathcal{QM}_{4n+4} into three classes:

- Ia)** The isometries associated with the Heisenberg algebra which exists in all cases, even when the internal Special Kähler manifold \mathcal{SK}_n has no continuous symmetries as it might happen when we deal the moduli space of a Calabi-Yau three-fold and the Heisenberg fields are the Ramond–Ramond scalars of a superstring compactification. The gauging of such perturbative isometries produces one field potentials that are of the pure exponential type.
- Ib)** The isometries of the inner Special Kähler manifold \mathcal{SK}_n that can always be promoted to isometries of the full Quaternionic manifold \mathcal{QM}_{4n+4} . Gauging these isometries one obtains a potential that at $\mathbf{Z}^\alpha = a = U = 0$ is identical with the potential obtained from the gauging of the same Kähler isometries in $\mathcal{N} = 1$ supersymmetry. Hence one would like to be able to stabilize these fields. As for $\mathbf{Z}^\alpha = a = 0$ there is no problem. We can always truncate them. The field U instead, that in superstring interpretations of the hypermultiplets can be identified with the coupling constant dilaton field, appears through exponentials all of the same sign in these perturbative gaugings and cannot be stabilized.
- Ic)** The non perturbative isometries that mix the Heisenberg symmetries with the Kähler symmetries. These exist only when \mathcal{QM}_{4n+4} is a symmetric homogeneous space and the Lie algebra of the full isometry group takes the universal form (4.53). Gauging a compact non perturbative generator appears to be the only way of introducing exponentials with opposite sign of the field U allowing for its stabilization.

II The Starobinsky-like potential:

$$V_{Starobinsky} = \text{const} \times \left(1 - \exp \left[-\sqrt{\frac{2}{3\alpha}} \phi \right] \right)^2 \quad (11.1)$$

can be obtained universally from all homogeneous symmetric Quaternionic Kähler manifolds by means of an admissible embedding of a $\mathfrak{sl}(2, \mathbb{R})$ Lie algebra into the Special Kähler subalgebra $\mathbb{U}_{SK} \subset \mathbb{U}_{\mathcal{Q}}$. The problem of embedding of $\mathfrak{sl}(2, \mathbb{R})$ algebras is the same as the problem of classifying standard triples and nilpotent algebras, yet the embedding must also be admissible, in the sense that it should allow for consistent one-field truncations. In the case of $\mathcal{SK} = \mathfrak{sp}(6, \mathbb{R})$ we exhausted the analysis of orbits and showed that admissible subalgebras correspond to embeddings of $\mathfrak{sl}(2)$ into the maximal subalgebra $(\mathfrak{sl}(2))^3$ yielding $\alpha = 1, \frac{2}{3}, \frac{1}{3}$. By means of arguments based on Tits Satake universality classes, we have advocated that this result is general for all supergravity theories where the hypermultiplets are described by a homogeneous symmetric space.

III The above results rely on the use of the minimal coupling Special Geometry for the description of the vector multiplets. Staying within such a framework one can gauge by different vector fields the generators of a maximal set of mutually commuting $\mathfrak{sl}(2, \mathbb{R})$ Lie algebras and obtain as many massive fields. The number of massive vector fields appears therefore to be equal to the rank of the Tits Satake subalgebra $\mathbb{G}_{\text{subTS}} \subset \mathbb{U}_{SK} \subset \mathbb{U}_{\mathcal{Q}}$. The result of [1] where it was found that the maximal number of massive vector fields in this sort of gauging is one is confirmed by this general rule. In [1] we have $\mathbb{G}_{\text{subTS}} = \mathfrak{sl}(2, \mathbb{R}) = \mathbb{U}_{SK} \subset \mathbb{U}_{\mathcal{Q}} \equiv \mathfrak{g}_{(2,2)}$ and $\text{rank } \mathfrak{sl}(2, \mathbb{R}) = 1$. In all other cases the rank of the sub Tits Satake Lie algebra is three.

PERSPECTIVES AND GENERALIZATIONS In order to improve our understanding of the possible embedding of inflaton dynamics within larger unified theories, we consider necessary and feasible to explore the following directions.

- A)** Enlarge the scope of gaugings of hypermultiplet isometries to non abelian algebras and consider the attractive situation where the Special Kähler manifold describing the vector multiplets and that, whose c -map provides the Quaternionic Kähler manifold description of the hypermultiplets are just the same or at least belong to the same class of homogeneous Special Kähler spaces.
- B** Utilize the above $\mathcal{N} = 2$ set up to promote the embedding of inflaton dynamics from $\mathcal{N} = 2$ to higher \mathcal{N} , in particular to $\mathcal{N} = 3$ and $\mathcal{N} = 4$.
- C** Consider such scenarios as superstring compactification on $T^2 \times K3$ and try to interpret the gaugings that produce inflaton dynamics in terms of fluxes.

We plan to address such questions in forthcoming publications.

Acknowledgements

We are grateful to our friends Sergio Ferrara and Antoine Van Proyen for useful discussions during the completion of this work. The work of A.S. was supported in part by the RFBR Grants No. 13-02-91330-NNIO-a, No. 13-02-90602-Arm-a and by the Heisenberg-Landau program.

A COSET REPRESENTATIVES AND OTHER LARGE FORMULAE NOT DISPLAYED IN THE MAIN TEXT

In this appendix we collect some large formulae that need the landscape format and are not presented in the main text.

A.1 CONCERNING THE $G_{(2,2)}$ MODEL

THE SOLVABLE COSET REPRESENTATIVE FOR $\frac{G_{(2,2)}}{SU(2) \times SU(2)}$

$$\mathbb{L} =$$

(A.1)

$$\begin{pmatrix} e^{\frac{h+U}{2}} & e^{\frac{U-h}{2}}y & -\sqrt{\frac{2}{3}}e^hZ_3 & -\frac{2Z_1}{\sqrt{3}} - \frac{2yZ_3}{\sqrt{3}} & -\sqrt{\frac{2}{3}}e^{-h}Z_3y^2 - 2\sqrt{\frac{2}{3}}e^{-h}Z_1y - \sqrt{2}e^{-h}Z_2 & e^{\frac{h-U}{2}}\left(a - \frac{Z_1Z_3}{3} - Z_2Z_4\right) & e^{-\frac{h}{2}-\frac{U}{2}}\left(\frac{2Z_2Z_3}{\sqrt{3}} - \frac{2Z_1^2}{3}\right) + e^{-\frac{h}{2}-\frac{U}{2}}y\left(a - \frac{Z_1Z_3}{3} - Z_2Z_4\right) \\ 0 & e^{\frac{U-h}{2}} & -\sqrt{2}e^hZ_4 & \frac{2Z_3}{\sqrt{3}} - 2yZ_4 & -\sqrt{2}e^{-h}Z_4y^2 + 2\sqrt{\frac{2}{3}}e^{-h}Z_3y + \sqrt{\frac{2}{3}}e^{-h}Z_1 & e^{\frac{h-U}{2}}\left(\frac{2Z_3^2}{3} + \frac{2Z_1Z_4}{\sqrt{3}}\right) & e^{-\frac{h}{2}-\frac{U}{2}}y\left(\frac{2Z_3^2}{3} + \frac{2Z_1Z_4}{\sqrt{3}}\right) + e^{-\frac{h}{2}-\frac{U}{2}}\left(a + \frac{Z_1Z_3}{3} + Z_2Z_4\right) \\ 0 & 0 & e^h & \sqrt{2}y & e^{-h}y^2 & -\sqrt{\frac{2}{3}}e^{\frac{h-U}{2}}Z_1 & -\sqrt{\frac{2}{3}}e^{-\frac{h}{2}-\frac{U}{2}}yZ_1 - \sqrt{2}e^{-\frac{h}{2}-\frac{U}{2}}Z_2 \\ 0 & 0 & 0 & 1 & \sqrt{2}e^{-h}y & \frac{2e^{\frac{h-U}{2}}Z_3}{\sqrt{3}} & \frac{2e^{-\frac{h}{2}-\frac{U}{2}}Z_1}{\sqrt{3}} + \frac{2e^{-\frac{h}{2}-\frac{U}{2}}yZ_3}{\sqrt{3}} \\ 0 & 0 & 0 & 0 & e^{-h} & \sqrt{2}e^{\frac{h-U}{2}}Z_4 & \sqrt{2}e^{-\frac{h}{2}-\frac{U}{2}}yZ_4 - \sqrt{\frac{2}{3}}e^{-\frac{h}{2}-\frac{U}{2}}Z_3 \\ 0 & 0 & 0 & 0 & 0 & e^{\frac{h-U}{2}} & e^{-\frac{h}{2}-\frac{U}{2}}y \\ 0 & 0 & 0 & 0 & 0 & 0 & e^{-\frac{h}{2}-\frac{U}{2}} \end{pmatrix} \quad (A.2)$$

A.2 EXPLICIT FORM OF GENERATORS FOR THE LIE ALGEBRA $\mathfrak{sp}(6, \mathbb{R})$ IN THE $\mathbf{14}'$ REPRESENTATION

The $\mathbf{14}'$ representation of $\mathfrak{sp}(6, \mathbb{R})$ which plays the role \mathbf{W} -representation for the special manifold under consideration is defined as the representation obeyed by the three-times antisymmetric tensors with vanishing \mathbb{C} -traces, namely:

$$\underbrace{t_{ABC}}_{\text{antisymmetric in } A, B, C} \times \mathbb{C}^{BC} = 0 \quad (A.3)$$

Let us then consider a lexicographic ordered basis for the 20-dimensional reducible representation provided by the three times antisymmetric tensor:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \\ V_7 \\ V_8 \\ V_9 \\ V_{10} \\ V_{11} \\ V_{12} \\ V_{13} \\ V_{14} \\ V_{15} \\ V_{16} \\ V_{17} \\ V_{18} \\ V_{19} \\ V_{20} \end{pmatrix} \equiv \begin{pmatrix} t_{123} \\ t_{124} \\ t_{134} \\ t_{234} \\ t_{125} \\ t_{135} \\ t_{235} \\ t_{145} \\ t_{245} \\ t_{345} \\ t_{126} \\ t_{136} \\ t_{236} \\ t_{146} \\ t_{246} \\ t_{346} \\ t_{156} \\ t_{256} \\ t_{356} \\ t_{456} \end{pmatrix} \quad (\text{A.4})$$

The splitting into the two irreducible subspaces of dimension **14** and **6** respectively can be performed by defining the following new basis vectors:

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \Phi_3 \\ \Phi_4 \\ \Phi_5 \\ \Phi_6 \\ \Phi_7 \\ \Phi_8 \\ \Phi_9 \\ \Phi_{10} \\ \Phi_{11} \\ \Phi_{12} \\ \Phi_{13} \\ \Phi_{14} \end{pmatrix} = \begin{pmatrix} V_1 \\ V_4 \\ V_6 \\ V_{10} \\ V_{11} \\ V_{15} \\ V_{17} \\ V_{20} \\ V_5 - V_{12} \\ -V_2 - V_{13} \\ V_7 - V_3 \\ V_{16} - V_9 \\ V_8 + V_{19} \\ V_{14} - V_{18} \end{pmatrix} ; \quad \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix} = \begin{pmatrix} V_5 + V_{12} \\ V_{13} - V_2 \\ -V_3 - V_7 \\ -V_9 - V_{16} \\ V_8 - V_{19} \\ V_{14} + V_{18} \end{pmatrix} \quad (\text{A.5})$$

8

Taking the antisymmetric cubic tensor product and using the splitting (A.5), the matrices $\widehat{\mathcal{D}}_{14}(\mathfrak{g})$ representing any element $\mathfrak{g} \in \mathfrak{sp}(6, \mathbb{R})$ of the Lie algebra in the **14'** representation can be easily extracted. The so obtained $\widehat{\mathcal{D}}_{14}(\mathfrak{g})$ matrices are symplectic,

since, by direct calculation one can determine a unique antisymmetric matrix:

$$\hat{\mathbb{C}}_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \quad (\text{A.6})$$

which verifies the relation:

$$\forall \mathfrak{g} \in \mathfrak{sp}(6, \mathbb{R}) \quad : \quad \hat{\mathcal{D}}_{14}(\mathfrak{g})^T \hat{\mathbb{C}}_{14} + \hat{\mathbb{C}}_{14} \hat{\mathcal{D}}_{14}(\mathfrak{g}) = 0 \quad (\text{A.7})$$

Unfortunately $\hat{\mathbb{C}}_{14}$ is not yet the standard symplectic matrix for the Lie algebra $\mathfrak{sp}(14, \mathbb{R})$. Hence we still need to perform a change of basis that brings $\hat{\mathbb{C}}_{14}$ to its standard form:

$$\mathbb{C}_{14} \equiv \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{A.8})$$

Such a change of basis is provided by the matrix:

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (\text{A.9})$$

Indeed we have:

$$\begin{aligned} \Lambda^T \widehat{\mathbb{C}}_{14} \Lambda^T &= \mathbb{C}_{14} \\ \mathcal{D}_{14}(\mathfrak{g}) &\equiv \Lambda^{-1} \widehat{\mathcal{D}}_{14}(\mathfrak{g}) \Lambda \\ 0 &= \mathcal{D}_{14}(\mathfrak{g})^T \mathbb{C}_{14} + \mathbb{C}_{14} \mathcal{D}_{14}(\mathfrak{g}) \end{aligned} \quad (\text{A.10})$$

We do not present the intermediate matrices $\widehat{\mathcal{D}}_{14}(\mathfrak{g})$. We go directly to the presentation of the final ones $\mathcal{D}_{14}(\mathfrak{g})$ that are standard symplectic matrices of the $\mathfrak{sp}(14, \mathbb{R})$ Lie algebra.

CARTAN GENERATORS IN THE $\mathbf{14}'$. The Cartan generators are named $\mathcal{H}_{14}^i \equiv \mathcal{D}_{14}(\mathcal{H}^i)$ and can be easily read-off from the following formula:

$$\sum_{i=1}^3 h_i \mathcal{H}_{14}^i =$$

$$\begin{pmatrix}
-h_1 - h_2 + h_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & h_1 - h_2 + h_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -h_1 + h_2 + h_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & h_1 + h_2 + h_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & h_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & h_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & h_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 + h_2 - h_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_1 + h_2 - h_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_1 - h_2 - h_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_1 - h_2 - h_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -h_3 & 0
\end{pmatrix} \quad (\text{A.11})$$

by collecting the coefficient of the parameter h_i .

POSITIVE ROOT STEP OPERATORS IN THE **14'** The step operator associated with the positive root α_i is named $\mathcal{E}_{14}^{\alpha_i} \equiv \mathcal{D}_{14}(\mathcal{E}^{\alpha_i})$ and can be easily read-off from the following formula:

92

$$\sum_{i=1}^9 a_i \mathcal{E}_{14}^{\alpha_i} =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}a_3 & 0 & 0 & 0 \\
\sqrt{2}a_9 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}a_1 & 0 & 0 & \sqrt{2}a_3 & 0 & 0 & -\sqrt{2}a_6 & 0 \\
-\sqrt{2}a_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}a_3 & 0 & 0 & -\sqrt{2}a_5 & 0 & 0 \\
0 & \sqrt{2}a_7 & -\sqrt{2}a_9 & 0 & \sqrt{2}a_5 & -\sqrt{2}a_6 & \sqrt{2}a_8 & \sqrt{2}a_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\sqrt{2}a_2 & 0 & 0 & 0 & a_1 & -a_4 & 0 & 0 & -\sqrt{2}a_5 & 0 & \sqrt{2}a_9 & a_8 & -a_6 \\
0 & 0 & -\sqrt{2}a_4 & 0 & 0 & 0 & a_2 & 0 & -\sqrt{2}a_6 & 0 & 0 & a_8 & \sqrt{2}a_7 & a_5 \\
-\sqrt{2}a_8 & 0 & \sqrt{2}a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -a_6 & a_5 & \sqrt{2}a_3 \\
0 & 0 & 0 & 0 & -\sqrt{2}a_2 & \sqrt{2}a_4 & 0 & 0 & -\sqrt{2}a_9 & \sqrt{2}a_7 & 0 & 0 & 0 & \sqrt{2}a_8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}a_7 & \sqrt{2}a_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}a_9 & 0 & \sqrt{2}a_4 & -\sqrt{2}a_1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{2}a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}a_5 & 0 & 0 & 0 \\
\sqrt{2}a_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}a_6 & -a_1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}a_1 & 0 & -\sqrt{2}a_8 & a_4 & -a_2 & 0
\end{pmatrix} \quad (\text{A.12})$$

by collecting the coefficient of the parameter a_i .

NEGATIVE ROOT STEP OPERATORS IN THE 14' The step operator associated with the negative root $-\alpha_i$ is named $\mathcal{E}_{14}^{-\alpha_i} \equiv \mathcal{D}_{14}(-\mathcal{E}^{\alpha_i})$ and can be easily read-off from the following formula:

$$\sum_{i=1}^9 b_i \mathcal{E}_{14}^{-\alpha_i} = \begin{pmatrix} 0 & \sqrt{2}b_9 & -\sqrt{2}b_7 & 0 & 0 & 0 & -\sqrt{2}b_8 & 0 & 0 & 0 & 0 & -\sqrt{2}b_2 & \sqrt{2}b_4 & 0 \\ 0 & 0 & 0 & \sqrt{2}b_7 & -\sqrt{2}b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2}b_9 & 0 & -\sqrt{2}b_4 & \sqrt{2}b_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}b_5 & 0 & 0 & 0 & -\sqrt{2}b_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\sqrt{2}b_6 & b_1 & 0 & 0 & \sqrt{2}b_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{2}b_1 & 0 & \sqrt{2}b_8 & -b_4 & b_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2}b_3 & 0 & 0 & -\sqrt{2}b_6 & 0 & -\sqrt{2}b_9 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}b_1 \\ 0 & \sqrt{2}b_3 & 0 & 0 & -\sqrt{2}b_5 & 0 & 0 & \sqrt{2}b_7 & 0 & 0 & 0 & 0 & 0 & 0 \\ \sqrt{2}b_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{2}b_7 & \sqrt{2}b_9 & 0 & -\sqrt{2}b_5 & \sqrt{2}b_6 & -\sqrt{2}b_8 \\ 0 & 0 & -\sqrt{2}b_5 & 0 & \sqrt{2}b_9 & b_8 & -b_6 & 0 & \sqrt{2}b_2 & 0 & 0 & 0 & -b_1 & b_4 \\ 0 & -\sqrt{2}b_6 & 0 & 0 & b_8 & \sqrt{2}b_7 & b_5 & 0 & 0 & \sqrt{2}b_4 & 0 & 0 & 0 & -b_2 \\ 0 & 0 & 0 & 0 & -b_6 & b_5 & \sqrt{2}b_3 & \sqrt{2}b_8 & 0 & -\sqrt{2}b_1 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{A.13})$$

by collecting the coefficient of the parameter b_i .

A.2.1 T

he explicit form of the metric in solvable coordinates reads:

$$\begin{aligned} ds_K^2 = & \, dh_1^2 + dh_2^2 + dh_3^2 + \frac{1}{2}e^{2h_2-2h_1}dp_1^2 + \frac{1}{2}e^{2h_3-2h_2}dp_2^2 \\ & + \frac{1}{2}e^{-4h_3}dp_3^2 + \frac{1}{2}e^{2h_3-2h_1}dp_4^2 + \frac{1}{2}e^{-2h_2-2h_3}dp_5^2 + \frac{1}{2}e^{-2h_1-2h_3}dp_6^2 + \frac{1}{2}e^{-4h_2}dp_7^2 \\ & + \frac{1}{2}e^{-2h_1-2h_2}dp_8^2 - \sqrt{2}e^{-2h_1-2h_2}dp_7p_1dp_8 + \frac{1}{2}e^{-4h_1}dp_9^2 - e^{2h_3-2h_1}dp_2dp_4p_1 - e^{-2h_1-2h_3}dp_5dp_6p_1 \\ & + \frac{1}{2}e^{2h_3-2h_1}dp_2^2p_1^2 + \frac{1}{2}e^{-2h_1-2h_3}dp_5^2p_1^2 + e^{-2h_1-2h_2}dp_7^2p_1^2 + e^{-4h_1}dp_8^2p_1^2 - \sqrt{2}e^{-4h_1}dp_8dp_9p_1 \\ & + \frac{1}{2}e^{-4h_1}dp_7^2p_1^4 - \sqrt{2}e^{-4h_1}dp_7dp_8p_1^3 + e^{-4h_1}dp_7dp_9p_1^2 - \sqrt{2}e^{-2h_2-2h_3}dp_3dp_5p_2 - \sqrt{2}e^{-4h_2}dp_5dp_7p_2 \\ & - e^{-2h_1-2h_2}dp_6dp_8p_2 + \sqrt{2}e^{-2h_1-2h_3}dp_3dp_6p_1p_2 + \sqrt{2}e^{-2h_1-2h_2}dp_6dp_7p_1p_2 + 2e^{-2h_1-2h_2}dp_5dp_8p_1p_2 \\ & + \sqrt{2}e^{-4h_1}dp_6dp_9p_1p_2 \\ & + \sqrt{2}e^{-4h_1}dp_6dp_7p_2p_1^3 - \sqrt{2}e^{-2h_1-2h_3}dp_3dp_5p_2p_1^2 - 2\sqrt{2}e^{-2h_1-2h_2}dp_5dp_7p_2p_1^2 \\ & - 2e^{-4h_1}dp_6dp_8p_2p_1^2 - \sqrt{2}e^{-4h_1}dp_5dp_9p_2p_1^2 \\ & - \sqrt{2}e^{-4h_1}dp_5dp_7p_2p_1^4 + \sqrt{2}e^{-4h_1}dp_6dp_7p_2p_1^3 + 2e^{-4h_1}dp_5dp_8p_2p_1^3 + e^{-2h_2-2h_3}dp_3^2p_2^2 \\ & + e^{-4h_2}dp_5^2p_2^2 + \frac{1}{2}e^{-2h_1-2h_2}dp_6^2p_2^2 \\ & + e^{-2h_1-2h_3}dp_3^2p_2^2 + 2e^{-2h_1-2h_2}dp_5^2p_2^2 + e^{-4h_2}dp_3dp_7p_2^2 - 2e^{-2h_1-2h_2}dp_5dp_6p_1p_2^2 - \sqrt{2}e^{-2h_1-2h_2}dp_3dp_8p_1p_2^2 \\ & - 2e^{-4h_1}dp_5dp_6p_2^2p_1^3 - \sqrt{2}e^{-4h_1}dp_3dp_8p_2^2p_1^3 + 2e^{-2h_1-2h_2}dp_5^2p_2^2p_1^2 + e^{-4h_1}dp_6^2p_2^2p_1^2 \end{aligned}$$

$$\begin{aligned}
& +2e^{-2h_1-2h_2}dp_3dp_7p_2^2p_1^2 + e^{-4h_1}dp_3dp_9p_2^2p_1^2 \\
& +e^{-4h_1}dp_5^2p_2^2p_1^4 + e^{-4h_1}dp_3dp_7p_2^2p_1^4 - 2\sqrt{2}e^{-2h_1-2h_2}dp_3dp_5p_2^3p_1^2 + \sqrt{2}e^{-2h_1-2h_2}dp_3dp_6p_2^3p_1 - \sqrt{2}e^{-4h_2}dp_3dp_5p_2^3 \\
& +\frac{1}{2}e^{-4h_1}dp_3^2p_2^4p_1^4 - \sqrt{2}e^{-4h_1}dp_3dp_5p_2^3p_1^4 + \sqrt{2}e^{-4h_1}dp_3dp_6p_2^3p_1^3 + e^{-2h_1-2h_2}dp_3^2p_2^4p_1^2 \\
& -2\sqrt{2}e^{-2h_1-2h_2}dp_3dp_5p_2^3p_1^2 + \frac{1}{2}e^{-4h_2}dp_3^2p_2^4 \\
& +\sqrt{2}e^{-2h_1-2h_3}dp_4dp_6p_3 - \sqrt{2}e^{-2h_1-2h_3}dp_4dp_5p_1p_3 \\
& -\sqrt{2}e^{-2h_1-2h_2}dp_4dp_8p_2p_3 + 2e^{-2h_1-2h_3}dp_3dp_4p_1p_2p_3 + 2e^{-2h_1-2h_2}dp_4dp_7p_1p_2p_3 \\
& +2e^{-4h_1}dp_4dp_7p_2p_3p_1^3 - 2\sqrt{2}e^{-4h_1}dp_4dp_8p_2p_3p_1^2 \\
& -2\sqrt{2}e^{-2h_1-2h_2}dp_4dp_5p_2^2p_3p_1 + 2e^{-4h_1}dp_4dp_9p_2p_3p_1 + \sqrt{2}e^{-2h_1-2h_2}dp_4dp_6p_2^2p_3 \\
& +2e^{-4h_1}dp_3dp_4p_2^3p_3p_1^3 - 2\sqrt{2}e^{-4h_1}dp_4dp_5p_2^2p_3p_1^3 + 2\sqrt{2}e^{-4h_1}dp_4dp_6p_2^2p_3p_1^2 \\
& +2e^{-2h_1-2h_2}dp_3dp_4p_2^3p_3p_1 + e^{-2h_1-2h_3}dp_4^2p_3^2 \\
& +e^{-2h_1-2h_2}dp_4^2p_2^2p_3^2 + 2e^{-4h_1}dp_4^2p_1^2p_2^2p_3^2 - e^{-2h_1-2h_2}dp_5dp_8p_4 - \sqrt{2}e^{-4h_1}dp_6dp_9p_4 + \sqrt{2}e^{-2h_1-2h_2}dp_5dp_7p_1p_4 \\
& +\sqrt{2}e^{-4h_1}dp_5dp_7p_4p_1^3 - \sqrt{2}e^{-4h_1}dp_6dp_7p_4p_1^2 - 2e^{-4h_1}dp_5dp_8p_4p_1^2 + 2e^{-4h_1}dp_6dp_8p_4p_1 + \sqrt{2}e^{-4h_1}dp_5dp_9p_4p_1 \\
& -2e^{-4h_1}dp_5^2p_2p_4p_1^3 + 4e^{-4h_1}dp_5dp_6p_2p_4p_1^2 - 2e^{-2h_1-2h_2}dp_5^2p_2p_4p_1 - 2e^{-4h_1}dp_6^2p_2p_4p_1 + e^{-2h_1-2h_2}dp_5dp_6p_2p_4 \\
& \sqrt{2}e^{-4h_1}dp_3dp_5p_2^2p_4p_1^3 - \sqrt{2}e^{-4h_1}dp_3dp_6p_2^2p_4p_1^2 + \sqrt{2}e^{-2h_1-2h_2}dp_3dp_5p_2^2p_4p_1 \\
& -2\sqrt{2}e^{-4h_1}dp_4dp_6p_2p_3p_4p_1 + \sqrt{2}e^{-2h_1-2h_2}dp_4dp_5p_2p_3p_4 \\
& +e^{-4h_1}dp_5^2p_4^2p_1^2 + 2\sqrt{2}e^{-4h_1}dp_4dp_5p_2p_3p_4p_1^2 - 2e^{-4h_1}dp_5dp_6p_4^2p_1 + \frac{1}{2}e^{-2h_1-2h_2}dp_5^2p_4^2 + e^{-4h_1}dp_6^2p_4^2
\end{aligned} \tag{A.14}$$

References

- [1] A. Ceresole, G. Dall’Agata, S. Ferrara, M. Trigiante and A. Van Proeyen *A Search for an $\mathcal{N} = 2$ Inflaton Potential*, arXiv:1404.1745 [hep-th].
- [2] S. Ferrara and S. Sabharwal, “Quaternionic Manifolds for Type II Superstring Vacua of Calabi-Yau Spaces,” Nucl. Phys. B **332** (1990) 317.
- [3] P. Frè , F. Gargiulo, J. Rosseel, K. Rulik, M. Trigiante and A. Van Proeyen, *Tits-Satake projections of homogeneous special geometries*,” Class. Quant. Grav. **24** (2007) 27-78. [hep-th/0606173].
- [4] P. Fre and A. S. Sorin, *Extremal Multicenter Black Holes: Nilpotent Orbits and Tits Satake Universality Classes*, JHEP **1301**, 003 (2013) [arXiv:1205.1233 [hep-th]]; P. Fre, A. S. Sorin and M. Trigiante, *Black Hole Nilpotent Orbits and Tits Satake Universality Classes*, arXiv:1107.5986 [hep-th].
- [5] R. Kallosh, A. Linde and D. Roest, *Superconformal Inflationary α -Attractors*, JHEP **1311** (2013) 198, arXiv:1311.0472 [hep-th]; R. Kallosh, A. Linde and D. Roest, *Large Field Inflation and Double α -Attractors*, arXiv:1405.3646 [hep-th]; R. Kallosh, A. Linde and D. Roest, *The double attractor behavior of induced inflation*, arXiv:1407.4471 [hep-th].
- [6] P. A. R. Ade *et al.* [Planck Collaboration], *Planck 2013 results. XXII. Constraints on inflation*, arXiv:1303.5082 [astro-ph.CO].
- [7] P. A. R. Ade *et al.* [Planck Collaboration], *Planck 2013 results. XVI. Cosmological parameters*, arXiv:1303.5076 [astro-ph.CO].
- [8] G. Hinshaw *et al.* [WMAP Collaboration], *Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Cosmological Parameter Results*, arXiv:1212.5226 [astro-ph.CO].
- [9] **BICEP2 Collaboration** , P. Ade *et al.*, *BICEP2 I: Detection Of B-mode polarization at degree angular scales*, arXiv:1403.3985 [astro-ph.CO]
- [10] A. A. Starobinsky, *A New Type of Isotropic Cosmological Models Without Singularity*, Phys. Lett. B **91** (1980) 99.
- [11] B. Whitt, “Fourth Order Gravity as General Relativity Plus Matter,” Phys. Lett. B **145** (1984) 176.
- [12] A. D. Linde, *A New Inflationary Universe Scenario: A Possible Solution of the Horizon, Flatness, Homogeneity, Isotropy and Primordial Monopole Problems*, Phys. Lett. B **108** (1982) 389; *Chaotic Inflation*, Phys. Lett. B **129** (1983) 177.
- [13] A. H. Guth, *The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems*, Phys. Rev. D **23** (1981) 347.
- [14] A. Albrecht and P. J. Steinhardt, *Cosmology for Grand Unified Theories with Radiatively Induced Symmetry Breaking*, Phys. Rev. Lett. **48** (1982) 1220.
- [15] For recent reviews see:
V. Mukhanov, “Physical foundations of Cosmology,” *Cambridge, UK: Univ. Pr. (2005)*;
S. Weinberg, “Cosmology,” *Oxford, UK: Oxford Univ. Pr. (2008)*.

- [16] P. G. Fre', *Gravity, a Geometrical Course*, Vol. 1, 2, Springer-Verlag (2012).
- [17] J. Ellis, D. Nanopoulos, K.A. Olive, *A No-Scale Supergravity Realization of the Starobinsky Model*, arXiv:1305.1247 [hep-th].
- [18] S. V. Ketov and A. A. Starobinsky, *Embedding $(R + R^2)$ -Inflation into Supergravity*, Phys. Rev. D **83** (2011) 063512, arXiv:1011.0240 [hep-th].
- [19] S. V. Ketov and A. A. Starobinsky, *Inflation and non-minimal scalar-curvature coupling in gravity and supergravity*, JCAP **1208** (2012) 022, arXiv:1203.0805 [hep-th].
- [20] S. V. Ketov, "Starobinsky Model in N=2 Supergravity," Phys. Rev. D **89** (2014) 085042 [arXiv:1402.0626 [hep-th]].
- [21] R. Kallosh and A. Linde, *Universality Class in Conformal Inflation*, JCAP **1307** (2013) 002, arXiv:1306.5220 [hep-th].
- [22] R. Kallosh and A. Linde, *Superconformal generalizations of the Starobinsky model*, JCAP **1306** (2013) 028, arXiv:1306.3214 [hep-th].
- [23] F. Farakos, A. Kehagias and A. Riotto, *On the Starobinsky Model of Inflation from Supergravity*, arXiv:1307.1137 [hep-th].
- [24] R. Kallosh and A. Linde, *Non-minimal Inflationary Attractors*, JCAP **1310** (2013) 033, arXiv:1307.7938 [hep-th].
- [25] R. Kallosh and A. Linde, *Multi-field Conformal Cosmological Attractors*, arXiv:1309.2015 [hep-th].
- [26] R. Kallosh, A. Linde and D. Roest, *A universal attractor for inflation at strong coupling*, arXiv:1310.3950 [hep-th].
- [27] S. Ferrara and M. Porrati, *Minimal $R + R^2$ Supergravity Models of Inflation Coupled to Matter*, arXiv:1407.6164 [hep-th].
- [28] P. Fré, A. Sagnotti and A. S. Sorin, *Integrable Scalar Cosmologies I. Foundations and links with String Theory*, Nucl.Phys. B **877** (2013) 1028-1106. arXiv:1307.1910 [hep-th].
- [29] P. Fré, A. S. Sorin and M. Trigiante, *Integrable Scalar Cosmologies II. Can they fit into Gauged Extended Supergravity or be encoded in $N=1$ superpotentials?*, Nucl. Phys. B **881** (2014) 91 [arXiv:1310.5340 [hep-th]].
- [30] S. Ferrara, R. Kallosh, A. Linde and M. Porrati, *Minimal Supergravity Models of Inflation*, Phys. Rev. D **88** (2013) 8, 085038 [arXiv:1307.7696 [hep-th]].
- [31] E. Cremmer, S. Ferrara, L. Girardello and A. Van Proeyen, *Yang-Mills Theories With Local Supersymmetry: Lagrangian, Transformation Laws And Superhiggs Effect*, Nucl. Phys. B **212** (1983) 413.
- [32] L. Castellani, R. D'Auria, P. Fré, *Supergravity and String Theory: a geometric perspective*, World Scientific, book in three volumes (1990).

- [33] P. Fre' and A. S. Sorin, *Inflation and Integrable one-field Cosmologies embedded in Rheonomic Supergravity*, Fortsch. Phys. **62** (2014) 4 [arXiv:1308.2332 [hep-th]].
- [34] P. Fre' and A. S. Sorin, *Axial Symmetric Kähler manifolds, the D-map of Inflaton Potentials and the Picard-Fuchs Equation*, Fortsch. Phys. **62** (2014) 26 [arXiv:1310.5278 [hep-th]].
- [35] A. Van Proeyen, *Massive Vector Multiplets in Supergravity*, Nucl. Phys. B **162** (1980) 376.
- [36] D. Z. Freedman, *Supergravity with Axial Gauge Invariance*, Phys. Rev. D **15** (1977) 1173.
- [37] S. Cecotti, S. Ferrara, M. Porrati and S. Sabharwal, *New Minimal Higher Derivative Supergravity Coupled To Matter*, Nucl. Phys. B **306** (1988) 160.
- [38] S. Cecotti, *Higher Derivative Supergravity Is Equivalent To Standard Supergravity Coupled To Matter. I*, Phys. Lett. B **190**, 86 (1987).
- [39] S. Ferrara, R. Kallosh and A. Van Proeyen, *On the Supersymmetric Completion of $R + R^2$ Gravity and Cosmology*, JHEP **1311** (2013) 134 [arXiv:1309.4052 [hep-th]].
- [40] S. Ferrara, R. Kallosh, A. Linde and M. Porrati, *Higher Order Corrections in Minimal Supergravity Models of Inflation*, arXiv:1309.1085 [hep-th]. To appear in JCAP.
- [41] S. Ferrara, P. Fre and A. S. Sorin, *On the Topology of the Inflaton Field in Minimal Supergravity Models*, JHEP **1404** (2014) 095, arXiv:1311.5059 [hep-th].
- [42] S. Ferrara, P. Fre and A. S. Sorin, *On the Gauged Kähler Isometry in Minimal Supergravity Models of Inflation,* Fortsch. Phys. **62** (2014) 277 [arXiv:1401.1201 [hep-th]].
- [43] W. Ballmann, M. Gromov, V. Schroeder, *Manifolds of Nonpositive Curvature*, Birkhäuser Boston-Basel-Stuttgart, 1985.
- [44] Mikhail Gromov, *Hyperbolic groups*. Essays in group theory. Math. Sci. Res. Inst. Publ. 8. New York (1987) Springer. pp. 75 – 263.
- [45] P. Fayet and J. Iliopoulos, *Spontaneously Broken Supergauge Symmetries and Goldstone Spinors*, Phys. Lett. B **51** (1974) 461.
- [46] P. Fré , M. Trigiante and A. Van Proeyen, *Stable de Sitter vacua from $N=2$ Supergravity* Class. Quant. Grav. **19** (2002) 4167 [arXiv:hep-th/0205119].
- [47] S. B. Giddings, S. Kachru and J. Polchinski, *Hierarchies from fluxes in string compactifications*, Phys. Rev. D **66** (2002) 106006 [arXiv:hep-th/0105097].
- [48] M. Gomez-Reino and C. A. Scrucca, *Locally stable non-supersymmetric Minkowski vacua in supergravity*, JHEP **0605** (2006) 015 [arXiv:hep-th/0602246]; O. Lebedev, H. P. Nilles and M. Ratz, *de Sitter vacua from matter superpotentials*, Phys. Lett. B **636** (2006) 126 [arXiv:hep-th/0603047]; E. Dudas, C. Papineau and S. Pokorski, *Moduli stabilization and uplifting with dynamically generated F -terms*, JHEP **0702** (2007) 028 [arXiv:hep-th/0610297]; H. Abe, T. Higaki, T. Kobayashi and Y. Omura, *Moduli stabilization, F -term uplifting and soft supersymmetry breaking terms*, Phys. Rev. D **75** (2007) 025019 [arXiv:hep-th/0611024].

- [49] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, *De Sitter vacua in string theory*, Phys. Rev. D **68** (2003) 046005 [arXiv:hep-th/0301240]; S. Kachru, R. Kallosh, A. D. Linde, J. M. Maldacena, L. P. McAllister and S. P. Trivedi, *Towards inflation in string theory* JCAP **0310** (2003) 013 [arXiv:hep-th/0308055].
- [50] F. Catino, C.A. Scrucca, P. Smyth, *Metastable de Sitter vacua in $N=2$ to $N=1$ truncated supergravity*, JHEP **1210** (2012) 124, [arXiv: hep-th/1209.0912v1].
- [51] E. Cremmer, S. Ferrara, C. Kounnas and D. V. Nanopoulos, *Naturally Vanishing Cosmological Constant In $N=1$ Supergravity*, Phys. Lett. B **133** (1983) 61.
- [52] *Pioneering papers on vector multiplet coupling to Supergravity:*
 B. de Wit, P. G. Lauwers, R. Philippe, S. Q. Su and A. Van Proeyen, *Gauge and Matter Fields Coupled to $N=2$ Supergravity*, Phys. Lett. B **134** (1984) 37;
 B. de Wit and A. Van Proeyen, *Potentials and Symmetries of General Gauged $N=2$ Supergravity: Yang-Mills Models*, Nucl. Phys. B **245** (1984) 89;
 E. Cremmer, C. Kounnas, A. Van Proeyen, J. P. Derendinger, S. Ferrara, B. de Wit and L. Girardello, *Vector Multiplets Coupled to $N=2$ Supergravity: SuperHiggs Effect, Flat Potentials and Geometric Structure*, Nucl. Phys. B **250** (1985) 385.;
- [53] *Definition of Special Kähler Geometry:*
 A. Strominger, *Special Geometry*, Comm. Math. Phys. **133** (1990) 163;
- [54] L. Castellani, R. D'Auria and S. Ferrara, *Special Kahler Geometry: An Intrinsic Formulation From $N = 2$ Space-time Supersymmetry*, Phys. Lett. B **241** (1990) 57; *Special geometry without special coordinates*, Class. Quant. Grav. **7** (1990) 1767.
- [55] R. D'Auria, S. Ferrara and P. Fre, *Special and quaternionic isometries: General couplings in $N=2$ supergravity and the scalar potential*, Nucl. Phys. B **359** (1991) 705.
- [56] E. Cremmer and A. Van Proeyen, *Classification of Kahler Manifolds in $N = 2$ Vector Multiplet Supergravity Couplings*, Class. Quant. Grav. **2** (1985) 445.
- [57] P. Fre, *Lectures on special Kahler geometry and electric - magnetic duality rotations*, Nucl. Phys. Proc. Suppl. **45BC** (1996) 59 [hep-th/9512043].
- [58] *Homogeneous Special Kähler Geometry:*
 B. de Wit, A. Van Proeyen, *Special geometry, cubic polynomials and homogeneous quaternionic spaces*, Commun.Math.Phys.149:307-334 (1992), hep-th/9112027;
 B. de Wit, A. Van Proeyen, *Broken sigma model isometries in very special geometry*, Phys.Lett.B293:94-99 (1992), hep-th/9207091;
- [59] D.V. Alekseevskii, Math. USSR Izvestija **9** (1975) 297.
- [60] S. Cecotti, "Homogeneous Kahler Manifolds and T Algebras in $N = 2$ Supergravity and Superstrings," Commun. Math. Phys. **124** (1989) 23.

- [61] B. de Wit, F. Vanderseypen, A. Van Proeyen, *Symmetry structure of special geometries*, Nucl.Phys. B400 (1993) 463-524, hep-th/9210068;
B. de Wit, A. Van Proeyen, *Hidden symmetries, special geometry and quaternionic manifolds*, Int.J.Mod.Phys. D3 (1994) 31-48, hep-th/9310067.
- [62] J. Bagger and E. Witten, *The Gauge Invariant Supersymmetric Nonlinear Sigma Model*, Phys. Lett. B **118** (1982) 103.
- [63] L. Andrianopoli, M. Bertolini, A. Ceresole, R. D'Auria, S. Ferrara, P. Fre and T. Magri, *$N = 2$ supergravity and $N = 2$ super Yang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the moment map*, J. Geom. Phys. **23** (1997) 111 [arXiv:hep-th/9605032].
- [64] N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, *HyperKähler Metrics and Supersymmetry*, Commun. Math. Phys. **108** (1987) 535.
- [65] K. Galicki, *A Generalization of the Momentum Mapping Construction for Quaternionic Kahler Manifolds*, Commun. Math. Phys. **108** (1987) 117.
- [66] A. Ceresole and G. Dall'Agata, *General Matter coupled $\mathcal{N} = 2$, $D = 5$ gauged supergravity* hep-th/0004111
- [67] R. D'Auria, S. Ferrara, M. Trigiante and S. Vaula, "Gauging the Heisenberg algebra of special quaternionic manifolds," Phys. Lett. B **610** (2005) 147 [hep-th/0410290]; R. D'Auria, S. Ferrara, M. Trigiante and S. Vaula, "Scalar potential for the gauged Heisenberg algebra and a non-polynomial antisymmetric tensor theory," Phys. Lett. B **610** (2005) 270 [hep-th/0412063]; R. D'Auria, S. Ferrara and M. Trigiante, "On the supergravity formulation of mirror symmetry in generalized Calabi-Yau manifolds," Nucl. Phys. B **780** (2007) 28 [hep-th/0701247 [HEP-TH]]; D. Cassani, S. Ferrara, A. Marrani, J. F. Morales and H. Samtleben, "A Special road to AdS vacua," JHEP **1002** (2010) 027 [arXiv:0911.2708 [hep-th]].
- [68] *Pioneering papers on the Solvable Lie algebra of cosets:*
L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fre, R. Minasian and M. Trigiante, *Solvable Lie algebras in type IIA, type IIB and M theories*, Nucl. Phys. B **493** (1997) 249 [hep-th/9612202];
P. Fre, L. Girardello, I. Pesando and M. Trigiante, *Partial $N=2 \rightarrow N=1$ local supersymmetry breaking and solvable Lie algebras*, hep-th/9611188;
L. Andrianopoli, R. D'Auria, S. Ferrara, P. Fre and M. Trigiante, *RR scalars, U duality and solvable Lie algebras*, Nucl. Phys. B **496** (1997) 617 [hep-th/9611014];
P. Fre, L. Girardello, I. Pesando and M. Trigiante, *Spontaneous $N=2 \rightarrow N=1$ local supersymmetry breaking with surviving compact gauge group*, Nucl. Phys. B **493** (1997) 231 [hep-th/9607032];
M. Trigiante, *Dualities in supergravity and solvable Lie algebras*, hep-th/9801144.
- [69] P. Fre', F. Gargiulo, K. Rulik and M. Trigiante, *The General pattern of Kac Moody extensions in supergravity and the issue of cosmic billiards*, Nucl. Phys. B **741** (2006) 42 [hep-th/0507249].
- [70] P. Fre, A. S. Sorin and , *Supergravity Black Holes and Billiards and Liouville integrable structure of dual Borel algebras*, JHEP **1003** (2010) 066 [arXiv:0903.2559 [hep-th]].
- [71] P. Frè , A. S. Sorin and M. Trigiante, *Integrability of Supergravity Black Holes and New Tensor Classifiers of Regular and Nilpotent Orbits*, JHEP **1204** (2012) 015, [arXiv:1103.0848 [hep-th]].

- [72] F. Cordaro, P. Fre, L. Gualtieri, P. Termonia and M. Trigiante, “N=8 gaugings revisited: An Exhaustive classification,” Nucl. Phys. B **532** (1998) 245 [hep-th/9804056].
- [73] H. Nicolai and H. Samtleben, “Compact and noncompact gauged maximal supergravities in three-dimensions,” JHEP **0104** (2001) 022 [hep-th/0103032].
- [74] B. de Wit, H. Samtleben and M. Trigiante, “On Lagrangians and gaugings of maximal supergravities,” Nucl. Phys. B **655** (2003) 93 [hep-th/0212239].
- [75] B. de Wit, H. Samtleben and M. Trigiante, “Magnetic charges in local field theory,” JHEP **0509** (2005) 016 [hep-th/0507289].
- [76] S. Ferrara, A. Marrani, E. Orazi and M. Trigiante, “Dualities Near the Horizon,” JHEP **1311** (2013) 056 [arXiv:1305.2057 [hep-th]].